Projectile Motion with Quadratic Damping in a Constant Gravitational Field

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Keywords
Divergent series, extrapolation methods, transforms.

The motion of a projectile in a constant gravitational field is an age-old problem. If there is no resistive force, the problem is simple. However, with the intoduction of resistive forces, the complexity of the problem increases. Recently, there has been a growing interest in the problem of projectile motion under a resistive force proportional to the square of velocity. The problem for resistive force proportional to vv cannot be solved in closed analytical form. A series solution of the equation leads to divergent series for $x(t)$ and $y(t)$ for reasonable values of the initial speed and angle of projection. However, we demonstrate that by inverting the series for $x(t)$, i.e., expressing t in terms of x, and substituting this in $y(t)$ one can express y in terms of x which is a highly convergent series and one can obtain the path of projectile with a few terms of the series. We further demonstrate that one can obtain the path even with a few terms of the divergent series for $x(t)$ and $y(t)$ by using extrapolation methods.

1. Introduction

The motion of a projectile (without resistance) in a constant gravitational field is a well discussed problem and is included in school and undergraduate text books. The study of projectile motion in a resisting medium is intimately connected with the study of ballistics and also finds its applications in various sports [1–5]. The problem of projectile motion with resistive force $\alpha\mathbf{v}$ has been considered by a number of workers [6–9]. The range of the projectile can be expressed in terms of the

Lambert W-function [10] which is defined by the equation $W(x)e^{W(x)} = x$, and is valid in the range $\left(-\frac{1}{e}, \infty\right)$. The path of the projectile is no longer symmetrical.

The motion, with resistance proportional to $v\mathbf{v}$, leads to a set of coupled equations and cannot be solved analytically. A number of papers including the quadratic drag have already been published [11–16]. However, these works are either numerical simulations or approximate analytical solution to the probelm. For example, Warburton *et al* obtained approximate analytic solutions of the problem in three different regions, namely, low angle, intermediate angle and large angle of projection. It may be mentioned that the approximations made to obtain an analytic solution leads to one important fact that the resistive force is not along the line of the direction of motion. If the resistive force is proportional to $\hat{i}v_x^2 + \hat{j}v_y^2$, v_x and v_y being the horizontal and vertical components of the velocity $\mathbf{v}$, the resistive force is not along the line of the direction of motion. However, the problem can be analytically solved.

The problem is really complex if the resistive force is proportional to $v\mathbf{v}$, where an analytic solution of the problem cannot be obtained. Given a differential equation and the boundary conditions, a series solution of the problem can always be obtained. A problem arises if the series obtained is a divergent one. For most of the reasonable initial conditions, the series obtained for the problem is divergent. Interestingly, if one tries a series solution for the problems discussed above, one obtains a divergent series. Fortunately, analytic solutions exist for these problems as metioned earlier. The question arises: Can we extract the answer to the problem from a divergent series obtained in these ways? In Section 2, we find a series solution of the problem for which the resistive force is proportional to $\hat{i}v_x^2 + \hat{j}v_y^2$. It is seen that the series for both $x\,(t)$ and $y\,(t)$ are divergent. An interesting finding is that even so, the series representing y as a function of x is convergent. This is achieved by inverting the solution of $x(t)$, i.e., expressing t as a function of x and substituting this in the expressing for $y(t)$. We further demonstrate that a solution for the problem can be obtained using extrapolation methods on these divergent series and discuss these in Section 3. Finally in Section 4, we consider the problem where the resistive force is proportional to $v\mathbf{v}$.

2. Series Solution of the Differential Equation

If the resistive force is proportional to $\hat{i}v_x^2 + \hat{j}v_y^2$, the equation of motion in component form is given by

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = -\alpha v_x^2 \quad \text{and} \quad \frac{\mathrm{d}v_y}{\mathrm{d}t} = -\alpha v_y^2 - g, \tag{1}$$

α being the constant of proportionality. Using the boundary conditions $v_x = v_{x0}$, $v_y = v_{y0}$ and $x = y = 0$ at $t = 0$, one gets

$$\begin{aligned} x &= \frac{1}{\alpha}\ln\left(1 + \alpha v_{x0} t\right) \\ y &= \frac{1}{\alpha}\left[\ln\cos\left(\gamma - \sqrt{\alpha g}\, t\right) - \ln\cos\gamma\right], \end{aligned} \tag{2}$$

where $\gamma = \tan^{-1}\left(\sqrt{\frac{\alpha}{g}} v_{y0}\right)$. The range of the projectile is given by

$$R = \frac{1}{\alpha}\ln\left(1 + 2v_{x0}\sqrt{\frac{\alpha}{g}}\tan^{-1}\left(\sqrt{\frac{\alpha}{g}}v_{y0}\right)\right). \tag{3}$$

The time of flight and the maximum height attained are given by

$$T = \frac{2}{\sqrt{\alpha g}}\tan^{-1}\left(\sqrt{\frac{\alpha}{g}}v_{y0}\right), \quad H = -\frac{1}{\alpha}\ln\cos\gamma. \tag{4}$$

A series solution for the problem is as follows:

$$
\begin{aligned}
x &= v_{x0}t - \frac{1}{2}\alpha(v_{x0}t)^2 + \frac{1}{3}\alpha^2(v_{x0}t)^3 - \frac{1}{4}\alpha^3(v_{x0}t)^4 + \cdots \\
y &= v_{y0}t - (g + \alpha v_{y0}^2)\frac{t^2}{2} + (\alpha g v_{y0} + \alpha v_{y0}^3)\frac{t^3}{3} \\
&\quad -(\alpha g^2 + 4\alpha^2 g v_{y0}^2 + 3\alpha^3 v_{y0}^4)\frac{t^4}{4} + \cdots . \qquad (5)
\end{aligned}
$$

The series for x can be easily identified to be that of $\frac{1}{\alpha}\ln(1 + \alpha v_{x0}t)$. However, it is almost impossible to guess the function from the series expansion for y. The series for $\ln(1+z)$ is covergent for $|z| \leq 1$ and is divergent for $|z| > 1$. Thus the series for x will diverge for most of the reasonable values of the parameters. The series for y will diverge at a faster rate than the series for x as is evident from (5). In order to obtain the equation of the path, one has to eliminate t from the above two equations. This is achieved by inverting the series for x and expressing t in terms of x and substituting this in the expressing for y. For the problem considered here, the inverse function is easily obtained to be $t = \frac{1}{\alpha v_{x0}}(e^{\alpha x} - 1)$, and y can be analytically expressed in terms of x. Note that the series for x as a function of t is divergent for $|\alpha v_{x0}t| > 1$; the inverse series is convergent for any value of the argument. For the values of the parameters $v_{x0} = v_{y0} = 10$ m/s, and $\alpha = 1\,\mathrm{m}^{-1}$, the series for x and y as functions of t, and y as a function of x are given by:

$$
\begin{aligned}
x(t) &= 10t - 50\,t^2 + 333.333\,t^3 - 2500\,t^4 + 20000.\,t^5 \\
&\quad - 166667.0\,t^6 + \cdots , \\
y(t) &= 10t - 54.9\,t^2 + 366.0\,t^3 - 2834.67\,t^4 + 23394.7\,t^5 \\
&\quad - 201168.0\,t^6 + \cdots . \\
y(x) &= x - 0.049\,x^2 - 0.0163333\,x^3 - 0.0130503\,x^4 \\
&\quad - 0.00440347\,x^5 - 0.00186495\,x^6 \\
&\quad - 0.000580352\,x^7 - 0.000209366\,x^8 - \cdots . \qquad (6)
\end{aligned}
$$

As the series for y as a function of x is convergent, the path of the projectile can be reproduced with a few terms of the series.

2.1 *Inversion of Series*

Even if the functional form of $x\,(t)$ is not known one can invert a series iteratively. For example, we can write the series for x given in (6) as follows:

$$t = 0.1x + 5\,t^2 - 33.333\,t^3 + 250\,t^4 - 2000\,t^5 + 16666.7\,t^6 + \cdots \,. \tag{7}$$

In the first approximation we assume that $t = 0.1x$ and replace t on the right-hand side of (7) by $0.1x$ and this gives the inverted series correct upto the second term which is $0.1x + 0.05x^2$. In the next approximation we replace t on the right-hand side of (7) by $0.1x + 0.05x^2$ and obtain the inverted series correct upto the third term and use this in the next approximation. Proceeding in this way, we obtain

$$\begin{aligned} t &= 0.1\,x + 0.05x^2 + 0.0166667\,x^3 + 0.00416667\,x^4 \\ &\quad + 0.00083333\,x^5 + \cdots \,. \end{aligned}$$

Substituting this in the expression for $y\,(t)$ in (6) we obtain y as a function of x and recover $y\,(x)$ given in (6).

3. Extrapolation Methods: Summing a Divergent Series

We have seen from (6) that a series solution of the equation gives divergent series for $x\,(t)$ and $y\,(t)$. In the next section, we shall consider the motion of a projectile with the resistive force proportional to $v\mathbf{v}$ which cannot be solved analytically and we have to make a series solution of the problem. Even if the series for $y(x)$ is convergent, the convergence may be very slow and one may need a large number of terms of the series to obtain the answer with reasonable accuracy. In the series solution of the differential equations, one has to evaluate the higher order derivatives and the evaluation of the successive terms becomes increasingly difficult. To obtain the convergent series for $y(x)$, one has to invert the series for $x(t)$ which is also a laborious job. The question that naturally arises is whether one can *sum* a divergent series or find the sum of a slowly convergent series with a few terms of the series. There exist a number of extrapolation methods [17–20] for the purpose. The most popular of these extrapolation methods is the Padé approximant [21]. If a function $f(x)$ is given by the series expansion

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots , \qquad (8)$$

then the $[N/M]$ Padé approximant of $f(z)$ is the uniquely determined rational polynomial defined by

$$[N/M]f(x) = \frac{A_N(z)}{B_M(z)}, \qquad (9)$$

where $A_N(z)$ and $B_M(z)$ are polynomials in z of degree N and M respectively, such that for any pair of integers (N, M)

$$f(z) - [N/M]f(x) = \mathcal{O}\left(z^{M+N+1}\right), \qquad z \to 0. \quad (10)$$

The basic idea of Padé approximation is to choose an approximating function in such a way that the value of the function at $z = 0$ and its first $N + M$ derivatives there agree with those of the given function and this is done by matching the first $N + M + 1$ terms of the original power series.

Since the value of a rational function remains unchanged if both the numerator and denominator are divided by the same constant, the constant term in $B_M(z)$ can be set to unity, so that the structure of the Padé approximant $[N/M]$ is

$$[N/M]f(x) = \frac{p_0 + p_1 z + p_2 z^2 + \ldots + p_N z^N}{1 + q_1 z + q_2 z^2 + \ldots + q_M z^M}, \qquad (11)$$

where the p_i's and the q_i's are constant coefficients and these are determined by using the condition imposed by (10). For $j = 0, 1, 2, \ldots, N + M$, this gives

$$\sum_{i=0}^{j} a_{j-i} q_i = p_j, \qquad j = 0, 1, 2 \ldots N; \quad p_j = 0 \text{ if } j > N\,,$$
$$\sum_{i=0}^{j} a_{j-i} q_i = 0, \qquad j = N+1, N+2 \ldots N+M\,, \quad (12)$$

with $q_j = 0$ if $j > M$ and $a_i = 0$ for $i < 0$. Equation (12) represents a set of $N + M + 1$ linear equations in $N + M + 1$ unknowns. This can be solved to get the coefficients $\{p_i\}$ and $\{q_i\}$ provided the coefficient matrix is non-singular. This determines the $N + M + 1$ constants of the approximant and substituting for these in (11), the approximant $[N/M]$ of $f(z)$ can be obtained. Of course, one can obtain the Padé approximants using the built-in function in *Mathematica*. The $[N/N]$ and $[N/N + 1]$ gives a bound for a class of functions. One great advantage of the Padé approximants is that it can simulate the poles of a function. However, it gives spurious poles in many situations.

One can construct approximants for the divergent series for $x(t)$ and $y(t)$ given in (6), evaluate x and y from the above approximants for different values of t and plot y versus x for different approximants. Such plots are shown in *Figure* 1. It is evident from the figure that [4/4] and [4/5] approximants, which need 8 and 9 terms of the series, give bounds for the function. The [5, 5] approximants reproduce the path with reasonable accuracy as can be seen from the figure. The [5,5] approximants for the divergent series for $x(t)$ and $y(t)$ given in (6) are:

$$[5/5]x(t) = \frac{10t + 200\,t^2 + 1305.56\,t^3 + 3055.56\,t^4 + 1812.17\,t^5}{1 + 25\,t + 222.22\,t^2 + 833.33\,t^3 + 1190.48\,t^4 + 396.825\,t^5}$$

$$[5/5]y(t) = \frac{10t + 170.324\,t^2 + 775.79\,t^3 + 317.928\,t^4 - 1921.4\,t^5}{1 + 22.5224\,t + 164.627\,t^2 + 394.742\,t^3 - 5.4688\,t^4 - 384.96\,t^5}$$

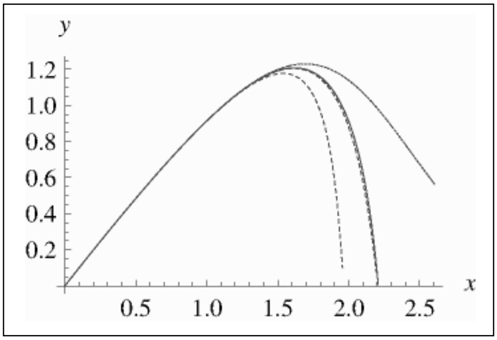

Figure 1. Plots of *y* versus *x* using the different different Padé approximants for *x*(*t*) and *y*(*t*) given by (6). The continuous curve is the exact path. The curve obtained with [4/4] approximants are denoted by dotted curve and are on the right of the exact path. The curve obtained with [4/5] approximants are denoted by dashed curve and are on the left. The curve obtained with [5/5] approximants is superposed on the exact curve.

The above approximants need 10 terms of the original series for their construction. One can find the time of flight of the projectile by equating the numerator of the approximant for $y(t)$ to zero. The only positive root of the equation is 0.80942 s which gives the time of flight. The range of the projectile is obtained by substituting this value of t in the approximant for $x(t)$ and this gives $R = 2.2067$ m; these are in close agreement with the exact values.

Recently it has been demonstrated that the Levin and Levin-like transforms are more powerful than the Padé approximants in a large class of problems [22–25]. To derive these transforms, we start with the *ansatz* (for a motivation see [21])

$$s = s_n + r_n = s_n + g_n \omega_n \ , \tag{13}$$

where $s_n = \sum_{i=0}^{n-1} a_n x^n$ is the partial sum of n terms of the series, s is the sum of the series, $\omega_n = \Delta s_n$ or Δs_{n-1}, Δ being the forward difference operator, i.e., $\Delta s_n = s_{n+1} - s_n$ and g_n is still unknown. It may be noted that Δs_n is the $(n+1)$-th term of the series; for $f(z)$ given by (8), $\Delta s_n = a_n z^n$. As s_n is the partial sum of n terms of the series, the term r_n can be interpreted as the remainder after n terms. An approximation for g_n provides a corresponding approximation for s. For example, if one assumes that r_n is a geometric series with the first term Δs_n and common ratio $\frac{\Delta s_{n+1}}{\Delta s_n}$, then one gets

$$s = s_n + \Delta s_n \left(\frac{1}{1 - \frac{\Delta s_{n+1}}{\Delta s_n}} \right) = s_n - \frac{(\Delta s_n)^2}{\Delta^2 s_n}\,, \qquad (14)$$

which gives the well-known Aitken's Δ^2-transform. The above assumption is equivalent to assuming that g_n is independent of n. Various models for g_n are assumed to derive different transforms. We consider a valid form for g_n

$$g_n = \sum_{i=0}^{\infty} \frac{\alpha_i}{(n+\beta)_i}\,, \qquad (15)$$

where α_i's are constant and $(z)_\nu$ is the Pochhammer symbol defined as

$$(z)_\nu = \frac{\Gamma(z+\nu)}{\Gamma(z)} = z(z+1)(z+2)\cdots(z+\nu-1)\,.$$

In (15), β is an arbitrary constant and has an important role to play. When the constants α_i's are known for a sequence, thereby its limit is also known. Therefore an approximation for the α's is equivalent to an approximation for s. Such an approximation for g_n may be obtained by terminating the infinite series in equation (15) at some value of i, i.e.,

$$g_{kn} = \sum_{i=0}^{k-1} \frac{\alpha_i}{(n+\beta)_i} = \frac{P_{k-1}}{(n+\beta)_{k-1}}\,,$$

where P_{k-1} is a polynomial in n of degree $k-1$ involving k constants. Thus

$$s = s_n + \frac{P_{k-1}}{(n+\beta)_{k-1}}\omega_n\,. \qquad (16)$$

As Δ^k will annihilate a polynomial of degree $k-1$, i.e.,

$$\Delta^k P_{k-1} = \Delta^k \left((n+\beta)_{k-1} \frac{(s-s_n)}{\omega_n} \right) = 0,$$

and since s is the limit of s_n as $n \to \infty$ and is independent of n, we have

$$s \approx \frac{\Delta^k \left((n+\beta)_{k-1} \frac{s_n}{\omega_n} \right)}{\Delta^k \left((n+\beta)_{k-1} \frac{1}{\omega_n} \right)} .$$

Using the expansion

$$\Delta^k f_n = \sum_{j=0}^{k} (-)^j \binom{k}{j} f_{n+j} ,$$

the explicit form for the transform (with $\omega_n = \Delta s_{n-1}$) is given by

$$\tau_k^{(n)} = \frac{\sum_{j=0}^{k} (-)^j \binom{k}{j} (n+\beta+j)_{k-1} \frac{s_{n+j}}{\Delta s_{n+j-1}}}{\sum_{j=0}^{k} (-)^j \binom{k}{j} (n+\beta+j)_{k-1} \frac{1}{\Delta s_{n+j-1}}} . \tag{17}$$

This is known as the τ-transform. The transform was introduced by Sidi [26]. Weniger [20] independently discovered it and fruitfully demonstrated that this can be an extremely useful computational tool. With $n = 1$, $\tau_k^{(n)} = \tau_k$ is given by

$$\tau_k = \frac{\sum_{j=0}^{k} (-)^j \binom{k}{j} (j+\beta+1)_{k-1} \frac{s_{j+1}}{\Delta s_j}}{\sum_{j=0}^{k} (-)^j \binom{k}{j} (j+\beta+1)_{k-1} \frac{1}{\Delta s_j}} . \tag{18}$$

If $f(z)$ is given by (8), then τ_k can be expressed as

$$\tau_k = \frac{\displaystyle\sum_{j=0}^{k} z^j \sum_{i=0}^{j} (-1)^i \binom{k}{i} (\beta+k-i+1)_{k-1} \frac{a_{j-i}}{a_{k-i}}}{\displaystyle\sum_{j=0}^{k} (-1)^j \binom{k}{j} (\beta+k-i+1)_{k-1} \frac{z^j}{a_{k-j}}} . \tag{19}$$

It can be easily seen that τ_k is also in the form of a ratio of two polynomials. The construction of the τ_k

needs $k+1$ terms of the original series. If we make a series expansion of τ_k in terms of the variable z, then at least $k+1$ terms of the expansion will agree with those of the original series for any value of the parameter β. Thus one can adjust the parameter β so as to obtain a better agreement with a given number of terms. Subsequently, we shall adjust β so that the $(k+2)$-th term of the series expansion of τ_k is exactly reproduced. The τ_5 approximant $(\beta = 0)$ for $x\,(t)$ given in (6) and the τ_5 approximant $(\beta = -3.35)$ for $y\,(t)$ are:

$$\begin{aligned}\tau_5[x(t)] &= \frac{10t + 181.481t^2 + 1027.78t^3 + 1909.17t^4 + 709.8766t^5}{1 + 23.1481t + 185.185t^2 + 595.238t^3 + 661.376t^4 + 132.275t^5}\\ \tau_5[y(t)] &= \frac{10t + 70.6588t^2 - 33.615t^3 - 44.8403t^4 - 47.3837t^5}{1 + 12.5559t + 28.9703t^2 - 21.5154t^3 + 36.5356t^4 - 57.135t^5}\end{aligned} \tag{20}$$

To construct the above approximants one needs only 6 terms of the original series. The τ_4 approximant $(\beta = -3.3)$ for the series $y\,(x)$ given in (6), which needs only five terms of the original series is:

$$\tau_4[y(x)] = \frac{x - 0.304346x^2 - 0.037825x^3 - 0.0129919x^4}{1 - 0.255346x - 0.0340036x^2 - 0.00577839x^3 - 0.00023258x^4} \tag{21}$$

A plot of the path obtained from the τ_5 approximants given above by evaluating the approximants at different values of t are shown by dots in *Figure* 2. It is seen from the figure that the τ_5 approximant reproduces fairly the function shown by the continuous curve. A plot of the τ_4 approximant is also shown in the figure by dotted line.

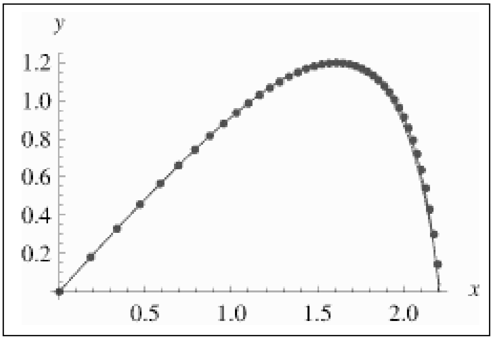

Figure 2. The dots are those obatined with the τ_5 approximants (equation (20)) for $x(t)$ and $y(t)$ given by (6). The dotted cuve is the plot of τ_4 approximant (equation (21)) for $y(x)$ and is inseparable from the exact curve shown by the continuous line.

One can calculate all the relevant quantities using these approximants. The range of the projectile can be obtained by equating the numerator of the approximant for $y(x)$ to zero which is a polynomial of order 4 in the present case. Two of the roots are complex, one root is zero and the other root is 2.214. Hence, the range obtained from the approximant is $R = 2.214$ m, the exact range being $R = 2.208$ m. The maximum height attained as obtained from the approximant is $H = 1.2044$ m and this occurs at $x = 1.6083$ m, the exact value being 1.2081 m. The time of flight can be obtained by equating the numerator of approximant for $y(t)$ to zero and solving for t. The τ_5 approximant for $y(t)$ gives $T = 0.8155$ s, the exact value being 0.8097 s. One can get the range by substituting this value of t in the approximant for $x(t)$ and this gives $R = 2.217$. We thus gain confidence that we can determine all the relevant quantities related to projectile motion from the approximants and proceed to obtain these for the projectile motion for which the resistive force is proportional to $v\mathbf{v}$.

4. Projectile Motion; Resistance Force Proportional to $v\mathbf{v}$

In this section we study the projectile motion with resistive force proportional $v\mathbf{v}$. As the resistive force per unit mass is $\alpha v\mathbf{v}$, it can be written as

$$\alpha v\mathbf{v} = \hat{i}\,\alpha v_x\sqrt{v_x^2 + v_y^2} + \hat{j}\,\alpha v_y\sqrt{v_x^2 + v_y^2},$$

and the equation of motion is given by

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = -\alpha v_x\sqrt{v_x^2 + v_y^2}, \quad \text{and} \quad \frac{\mathrm{d}v_y}{\mathrm{d}t} = -\alpha v_y\sqrt{v_x^2 + v_y^2} - g, \tag{22}$$

with the boundary conditions $v_x = v_{x0}$, $v_y = v_{y0}$ and $x = y = 0$ at $t = 0$. Unlike the other cases considered in the previous sections, the equations are coupled. By a change of the variables, the equations may obtain a somewhat simpler form. For example, using the variables $\frac{1}{v_x}$ and $\frac{v_y}{v_x}$, the equations take a simpler form. If we set $p\,(t) = \frac{1}{v_x}$, $q\,(t) = \frac{v_y}{v_x}$, then the equations become

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \alpha\sqrt{1+q^2}, \quad \text{and} \quad \frac{\mathrm{d}q}{\mathrm{d}t} = -g\,p, \qquad (23)$$

with the boundary conditions $p = p_0 = \frac{1}{v_{x0}}$ and $q = q_0 = \frac{v_{y0}}{v_{x0}}$ at $t = 0$. The equations are still coupled. However, to find a series solution of the equation, this set of equations is more convenient and it takes little time to compute the derivatives.

The power series solution of the set of equations given in (23) can be written as

$$\begin{aligned} p\,(t) &= a_0 + a_1\,t + \frac{a_2}{2!}\,t^2 + \frac{a_3}{3!}\,t^3 + \frac{a_4}{4!}\,t^4 + \cdots\,, \\ q\,(t) &= b_0 + b_1\,t + \frac{b_2}{2!}\,t^2 + \frac{b_3}{3!}\,t^3 + \frac{b_4}{4!}\,t^4 + \cdots\,, \end{aligned} \qquad (24)$$

where $a_0 = p_0$, $b_0 = q_0$, $a_n = \left(\frac{\mathrm{d}p}{\mathrm{d}t}\right)_{t=0}$ and $b_n = \left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)_{t=0}$. It is evident that the expressions for a_n and b_n become more and more lengthy as n increases. Reciprocating the series for $p\,(t)$, one obtains the series for v_x and integrating the series term by term with respect to t one obtains the series for x as a function of t. Dividing the series for $q\,(t)$ by the series for $p\,(t)$, one obtains the series for v_y and integrating the series one obtains the series for y.

With $\alpha = 0.1\,m^{-1}$, $v_{x0} = 3$ m/s , $v_{y0} = 10$ m/s and $g = 9.8$ m/s^2 (angle of projection 73.3$^{\mathrm{o}}$), the series for x and y are:

$$\begin{aligned} x\,(t) = \; & 3\,t - 1.566046\,t^2 + 1.559335\,t^3 - 1.475489\,t^4 \\ & + 1.543993\,t^5 - 1.670439\,t^6 + 1.853274\,t^7 \\ & - 2.108707\,t^8 + 2.426999\,t^9 - 2.837955\,t^{10} \\ & + 3.343345\,t^{11} - 3.979082\,t^{12} + \cdots \end{aligned} \qquad (25)$$

$$y(t) = 10\,t - 10.120153\,t^2 + 6.903033\,t^3 - 7.020128\,t^4 + 7.160853\,t^5 - 7.777640\,t^6 + 8.636168\,t^7 - 9.805960\,t^8 + 11.308720\,t^9 - 13.201805\,t^{10} + 15.572572\,t^{11} - 18.516215\,t^{12} + \cdots \quad (26)$$

If we invert the series for $x(t)$ and express t as a function x, we obtain

$$t = \frac{x}{3} + 0.0580017\,x^2 + 9.431373 \times 10^{-4}\,x^3 - 1.896117 \times 10^{-3}\,x^4 - 4.055294 \times 10^{-4}\,x^5 + 7.541998 \times 10^{-6}\,x^6 + 2.480584 \times 10^{-5}\,x^7 + 6.603575 \times 10^{-6}\,x^8 + 7.228171 \times 10^{-7}\,x^9 + 2.932196 \times 10^{-9}\,x^{10} + 3.578439 \times 10^{-8}\,x^{11} + 3.811996 \times 10^{-8}\,x^{12} + \cdots$$

Substituting this in the expression for $y(t)$, one gets y as a function of x:

$$y(x) = \frac{10}{3}x - 0.544444\,x^2 - 0.126315\,x^3 - 0.0125155\,x^4 + 0.00215901\,x^5 + 0.00101383\,x^6 + 0.000112646\,x^7 - 3.017331 \times 10^{-5}\,x^8 - 1.449798 \times 10^{-5}\,x^9 - 2.285758 \times 10^{-6}\,x^{10} + 8.842523 \times 10^{-8}\,x^{11} + 8.892665 \times 10^{-8}\,x^{12} - 6.234862 \times 10^{-9}\,x^{13} - \cdots \quad (27)$$

A plot of the path with 12 terms of the series is shown in *Figure* 3. As can be seen from the figure, this exactly reproduces the path obtained numerically by Runge–Kutta method. The figure also shows the path with no air resistance. If we take 12 terms of the above series for $y(x)$, equate it to zero and solve for x, we get the range of the projectile as $R = 3.3857$ m. The maximum of the $y(x)$ (with 12 terms of the series) occurs at $x = 1.8314$ m and the maximum value of $y(x)$, which is the maximum height attained is $H = 3.4445$ m.

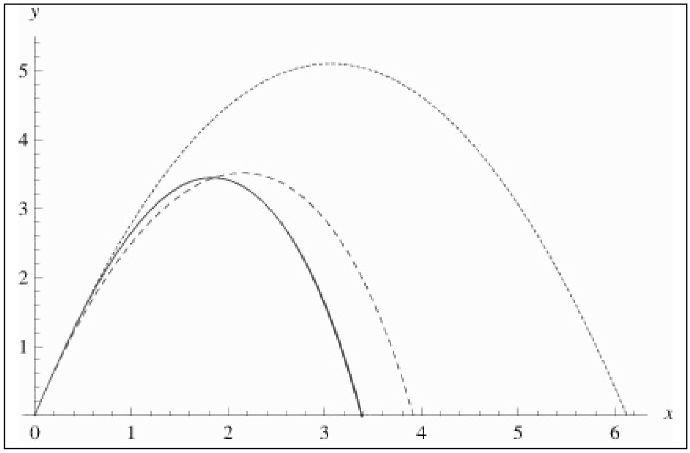

Figure 3. The dotted curve is for no air resistance. The exact curve (obtaibed numerically by Runge–Kutta method) for $\alpha = 0.1\ m^{-1}$, $v_{x0} = 3$ m/s, v_{y0}=10.0 m/s and $g = 9.8$ m/s^2 is shown by continuous line. The curve with twelve terms of the series given by (27) is shown by dotted curve and superposed on the continuous curve. The dashed curve is for reistance proportional to $\hat{i}\ v_x^2 + \hat{j}\ v_y^2$ with the same set of parameters.

To obtain an agreement with the exact curve, one needs a comparatively larger number of terms of the series for $y\,(x)$. In such cases, one can construct an approximant with a small number of terms of the series. The τ_5 approximant with 6 terms of the series for $y\,(x)$ is given by

$$\tau_5 = \frac{\frac{10}{3}x - 0.697903x^2 - 0.0854352x^3 - 0.00783692x^4 + 0.0023533x^5}{1 - 0.046037x + 0.0047444x^2 + 0.0004339x^3 + 0.00013610x^4 + 0.0002178x^5}$$

The value of the range obtained by equating the numerator of the above approximant to zero is $R = 3.3825$m which is in good agreement with the value obtained with 12 terms of the series for $y\,(x)$. Differentiating the expression for τ_5 with respect to x and equating the numerator to zero, one gets the value of x for which y is maximum and the value obtained is 1.8191 m. Substituting this value of x in the expression for τ_5, one gets $H = 3.4358$ m which is also in good agreement with the value obtained previously.

To find the time of flight, we construct the τ_5 approximants for $x\,(t)$ and $y\,(t)$ using 6 terms of the series. These are

$$\tau_5[x(t)] = \frac{3t + 7.36871t^2 + 6.12648t^3 + 2.02093t^4 + 0.619985t^5}{1 + 2.97825t + 3.07708t^2 + 1.22372t^3 + 0.196196t^4 + 0.00376227t^5}$$

$$\tau_5[y(t)] = \frac{10t + 0.914226t^2 - 4.15197t^3 + 0.446666t^4 - 0.45838t^5}{1 + 1.10344t + 0.0111996t^2 - 0.00369596t^3 + 0.00123556t^4 - 0.0007306t^5}$$

θ		2	5	8	12	16	20	26	33	40	49
R	(a)	91	183	274	366	457	549	640	732	823	914
	(b)	76	177	264	367	456	535	640	738	835	923
	(c)	77	182	277	390	491	582	705	829	941	1065
H	(a)	0.02	0.1	0.19	0.33	0.61	1.0	1.5	2.3	3.2	4.5
	(b)	0.011	0.07	0.17	0.36	0.62	0.93	1.5	2.3	3.2	4.5
	(c)	0.012	0.07	0.19	0.29	0.75	1.17	1.97	3.17	4.7	6.9
T	(b)	0.096	0.24	0.37	0.54	0.71	0.87	1.11	1.35	1.60	1.89
	(c)	0.098	0.24	0.39	0.59	0.78	0.98	1.27	1.61	1.95	2.38
T_m	(b)	0.48	0.12	0.18	0.26	0.34	0.42	0.52	0.63	0.74	0.87
	(c)	0.49	0.12	0.20	0.29	0.39	0.49	0.63	0.80	0.97	1.19
x_m	(b)	39	91	138	195	246	292	353	416	471	534
	(c)	39	96	149	215	277	355	416	503	582	674

Table 1. Different parameters of the projectile motion in air for small times of flight. The angle of projection θ is in minutes, range R in m, maximum height H in m, time of flight T in s, time to reach the maximum height T_m in s and x_m (in m) is the value of x for which the height is maximum. The experimental values are in the rows denoted by (*a*). The rows (*b*) show the calculated values, for resistance proportional to $v\mathbf{v}$, with τ_4 approximants using five terms of the series for $x(t)$ and y(t). The rows (*c*) show the calculated values, for resistance proportional to $\hat{i}\, v_x^2 + \hat{j}\, v_y^2$.

The time of flight obtained by equating the numerator of $\tau_5\,[y\,(t)]$ to zero and solving for t is $T = 1.579$ s. If we substitute this value of t in the expression for $\tau_5\,[x\,(t)]$, we get the range $R = 3.3873$ m which is in close agreement with the value of the range obtained earlier from the expression for $y\,(x)$. The path of the projectile can be obtained by evaluating $\tau_5\,[x\,(t)]$ and $\tau_5\,[y\,(t)]$ for different values of t and reproduces the exact path obtained numerically.

We now compare the experimental values [27] of different parameters of the projectile motion in air for small times of flight with the theoretical results. Kemp [27] considered projectiles of mass $m = 9.7$ g, diameter $d = 0.76$ cm and muzzle velocity 823 m/s and we consider the drag coefficient $\alpha = 1.05 \times 10^{-3}$ as recommended by the author.

Table 1 shows the different parameters of the projectile obtained theoretically along with the experimental values which are in the rows indicated by (a). In row (b) we place the values of the different parameters obtained as explained in the present section. To be more explicit, we have calculated 5 terms of the series for $x\,(t)$ and $y\,(t)$for each angle of projection and have constructed

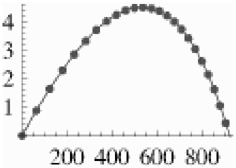

Figure 4. Exact path for muzzle velocity 823 m/s, θ = 49′ and α = 0.00105 m^{-1} obtained numerically by Runge–Kutta method is shown by the continuous line. The dots represent points obtained from the τ_4-approximants given by (28).

τ_4-approximants for these. For example, for angle of projection $\theta = 49'$ (shown in the last column of the table), the τ_4-approximants for $x(t)$ and $y(t)$ are

$$\tau_4[x(t)] = \frac{822.916\,t + 581.517\,t^2 + 44.409\,t^3 - 2.09255\,t^4}{1 + 1.13873\,t + 0.297039\,t^2 + 0.00368044\,t^3 - 0.00017038\,t^4}$$
$$\tau_4[y(t)] = \frac{11.7303\,t + 3.38932\,t^2 - 3.53532\,t^3 - 0.803589\,t^4}{1 + 1.13874\,t + 0.29704\,t^2 + 0.00277584\,t^3 - 0.00025274\,t^4} \qquad (28)$$

As can be seen from *Figure* 4 the τ_4-approximants almost exactly reproduce the path of the projectile. We calculate the different parameters of the projectile shown in the table as follows:

The time of flight is obtained by equating the numerator of $\tau_4[y(t)]$ to zero and solving for t. The two positive roots are 0 and 1.8944 and hence $T = 1.8944$ s. Substituting this value of t in the expression for $\tau_4[x(t)]$, we get $R = 923.4$ m. Time T_H to reach the maximum height is obtained by differentiating the expression for $\tau_4[y(t)]$ with respect to t and after simplification equating the numerator to zero and solving for t. The only positive root is $t = 0.87$ and hence $T_H = 0.87$ s. Substituting this value of t in $\tau_4[y(t)]$ one obtains the maximum height reached which in this case is $H = 4.59$ m. Substituting $t = T_H$ in the expression for $\tau_4[x(t)]$, one gets $x_H = 534$m.

It is evident from *Table* 1 that the range R and the maximum height H attained by the projectile, obtained by assuming that the resistive force is proportional to $v\mathbf{v}$ and using the extrapolation methods with only five terms of the series for $x(t)$ and $y(t)$, agree fairly with the experimental values, though for small angles the deviation is somewhat larger. It may be noted that the model in which the resistive force is assumed to be proportional to $\hat{i}v_x^2 + \hat{j}v_y^2$ is also a fair approximation for the problem.

It may be remarked that if one is interested in the range R and maximum height H attained by the projectile,

θ	$y(x)$
2	$0.000581776\,x - 7.234301 \times 10^{-6}x^2 - 5.064011 \times 10^{-9}x^3 - 2.658596 \times 10^{-12}x^4 - 1.116650 \times 10^{-15}x^5 - 3.909279 \times 10^{-18}x^6$
5	$0.000145444\,x - 7.234301 \times 10^{-6}x^2 - 5.064008 \times 10^{-9}x^3 - 2.658611 \times 10^{-12}x^4 - 1.116665 \times 10^{-15}x^5 - 3.909395 \times 10^{-18}x^6$
8	$0.00232711\,x - 7.234337 \times 10^{-6}x^2 - 5.064050 \times 10^{-9}x^3 - 2.658591 \times 10^{-12}x^4 - 1.116610 \times 10^{-15}x^5 - 3.908822 \times 10^{-18}x^6$
12	$0.00349067\,x - 7.234386 \times 10^{-6}x^2 - 5.064101 \times 10^{-9}x^3 - 2.658605 \times 10^{-12}x^4 - 1.116594 \times 10^{-15}x^5 - 3.908556 \times 10^{-18}x^6$
16	$0.00465424\,x - 7.234455 \times 10^{-6}x^2 - 5.064173 \times 10^{-9}x^3 - 2.658635 \times 10^{-12}x^4 - 1.116584 \times 10^{-15}x^5 - 3.908323 \times 10^{-18}x^6$
20	$0.00581783\,x - 7.234543 \times 10^{-6}x^2 - 5.064266 \times 10^{-9}x^3 - 2.658678 \times 10^{-12}x^4 - 1.116582 \times 10^{-15}x^5 - 3.908121 \times 10^{-18}x^6$
26	$0.00756324\,x - 7.234712 \times 10^{-6}x^2 - 5.064443 \times 10^{-9}x^3 - 2.658770 \times 10^{-12}x^4 - 1.116594 \times 10^{-15}x^5 - 3.907878 \times 10^{-18}x^6$
33	$0.00979961\,x - 7.234965 \times 10^{-6}x^2 - 5.064709 \times 10^{-9}x^3 - 2.658919 \times 10^{-12}x^4 - 1.116629 \times 10^{-15}x^5 - 3.907684 \times 10^{-18}x^6$
40	$0.0116361\,x - 7.235278 \times 10^{-6}x^2 - 5.065037 \times 10^{-9}x^3 - 2.659111 \times 10^{-12}x^4 - 1.116687 \times 10^{-15}x^5 - 3.907588 \times 10^{-18}x^6$
49	$0.0142545\,x - 7.235768 \times 10^{-6}x^2 - 5.065552 \times 10^{-9}x^3 - 2.659424 \times 10^{-12}x^4 - 1.116795 \times 10^{-15}x^5 - 3.907607 \times 10^{-18}x^6$

Table 2. Convergent series for $y(x)$ for different angles of projection θ (in minutes) for muzzle velocity 823 m/s and $\alpha = 0.00105$ m^{-1} obtained by inverting the series for $x(t)$ and substituting in $y(t)$ as explained in the text.

one can find it more easily from the convergent series $y(x)$. For ready reference, we give the expressions for the convergent series for $y(x)$ for the different angles of projection considered above in *Table* 2. As the series are highly convergent, a few terms of the series are sufficient to obtain R and H with reasonable accuracy.

Conclusions

The solution for the projectile motion with resistance proportional to $v\mathbf{v}$ cannot be obtained in closed analytical form. However, given the initial conditions, a series solution of the differential equation can always be made and obtain x and y as a power series in t. In most situations these series are divergent or slowly convergent. We demonstrate that it is possible to extract the relevant physical quantities even from these divergent or slowly convergent series by using extrapolation methods. The most interesting finding is that if we invert the series for $x(t)$ to obtain t as a function of x and substitute in $y(t)$, we obatin a highly convergent series for $y(x)$ and a few terms of the series can be used to obtain the path of the projectile. A closely related question that remains open is that, in reality, the air resistance contains both linear and quadratic terms. A complete analytical solution to the problem is not possible. However, a semi-analytic solution to the problem can be obtained in the way outlined in the text.

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