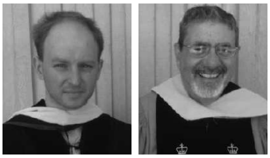
Why Some Pool Shots are More Difficult Than Others

Justin Jankunas and Richard N Zare



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The physics behind the game of billiards is rather well understood as is our grasp of classical mechanics. We present here a mathematical explanation of why slice shots are more difficult than direct shots. Despite a large number of treatises dedicated to the study of physics of billiards, it appears that the simple explanation has escaped our attention until now. We show that high impact-parameter shots impart a larger angular spread to the object ball, compared to head-on shots. The effect can be understood in terms of a non-linear relationship between the impact parameter and the scattering angle, and the fact that a real-world pool player does not have a perfect cue ball control; in other words, the impact parameter distribution is not a delta function, but has a finite spread. To keep the mathematics simple and not to obscure the underlying physical principles our treatment ignores the ball's spin, friction, and other well-known effects in the game of pool.

Anyone who has ever played a game of pool, a form of billiards, is intuitively familiar with the following scenario: When a cue ball, an object ball, and a pocket lie more or less in a straight line (*Figure* 1a), the shot is said to be much easier than the configuration in which a cue ball, an object ball, and a pocket are roughly at right angles (*Figure* 1b). Why is that? A consideration of this problem shows us how the use of straightforward classical mechanics instructs us about collision dynamics.

Keywords
Billiards, hard-sphere collisions, 2D vs. 3D scattering, differential cross-section.

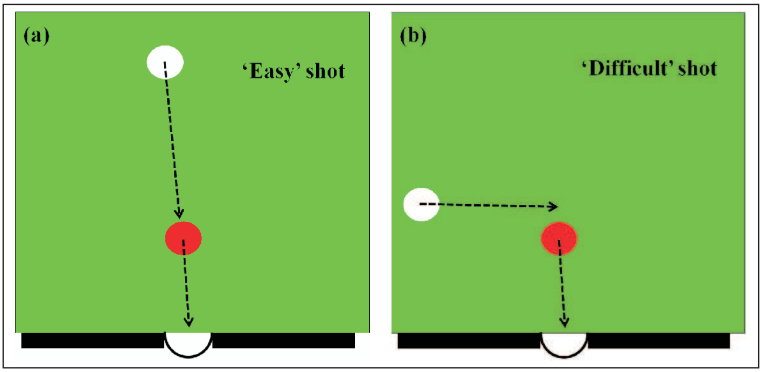


Figure 1. Comparison between **(a)** an 'easy' shot and **(b)** a 'difficult' shot in the game of pool. In the easy shot the cue ball (white), the object ball (red), and the pocket lie more or less in a straight line. In a difficult shot the cue ball, the object ball, and the pocket are nearly at right angles to each other. The same considerations apply to the board game called Carroms (Karroms). The game is very popular throughout South Asia mainly India, Pakistan, Bangladesh, Sri Lanka, Nepal, etc., and has gained some popularity in Europe and the United States where it has been introduced by the Indian diaspora.

In particle scattering theory, which describes molecular or particle collisions, one of the easiest problems to solve is the so-called hard-sphere collisions. Billiard ball collisions are good examples of impenetrable sphere scattering. When a cue ball misses an object ball, the cue ball continues in a straight line, but the object ball remains at rest. In other words, the two balls do not interact and the potential is said to be zero. When the two balls collide, to a good approximation, they do so in a completely elastic manner. This means that kinetic energy is conserved in the collision. In this case, the potential is said to be infinitely large. Mathematically, this is shown as $V(b) = 0$ for $b > d$ and $V(b) = \infty$ for $b \leq d$, where b is the familiar impact parameter that describes the distance of closest approach to the center of the object ball, if the cue ball were to follow a straight line path. Here d is the diameter of a billiard ball. The impact parameter b is taken as always positive. These two scattering parameters as well as others are shown in *Figure* 2.

Here the cue ball impacts the target ball at the impact parameter b and bounces elastically from it so that the angle of incidence α equals the angle of reflection. It is clear from the figure that $\sin\alpha = b/d$. The scattering angle θ in the center-of-mass frame (COM) is simply[1] $\theta = \pi - 2\alpha$. It follows that

$$\theta = \pi - 2\sin^{-1}\left(\frac{b}{d}\right), \tag{1a}$$

[1] The ensuing discussion pertains to the cue ball, and hence the scattering angle θ refers "to the angle through which the cue ball is scattered. The conclusions reached are identical to those if the trajectory and scattering angle of the object ball are followed, instead of the cue ball.

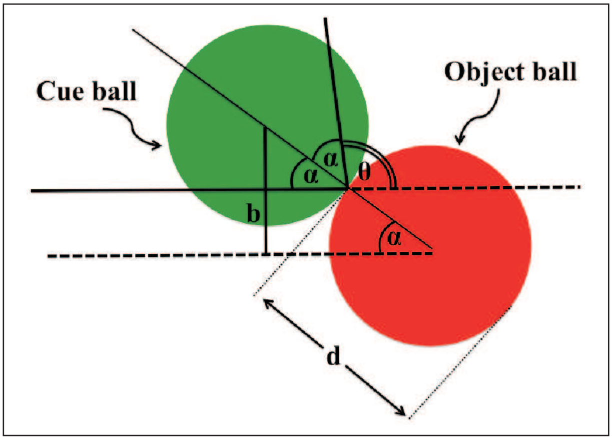


Figure 2. Typical hard sphere scattering diagram: b is the impact parameter, d is the diameter of a billiard ball, α is the angle of incidence and reflection, and θ is the scattering angle in the center-of-mass frame.

which may be rearranged to express the impact parameter as

$$b = d \sin\left(\frac{\theta}{2} - \frac{\pi}{2}\right) = d \cos\left(\frac{\theta}{2}\right) . \qquad (1b)$$

Equations (1a) and (1b) make up the key formulae for hard-sphere scattering. The answer to the question posed at the start of this article – why 'slice' or 'cut' shots are more difficult than direct shots – ultimately follows from these key formulae.

Nevertheless, as is evident from (1a), there is a one-to-one relationship between the scattering angle and the impact parameter. From this point of view, head-on and grazing collisions might be thought to be of equal 'difficulty'. Mathematically, this requires that the impact parameter distribution be a delta function, i.e., $P(b) = \delta(b)$.

However, this impact parameter distribution is never realized in practice. Even an expert pool player will have a finite spread in the distribution of impact parameters. In other words, all pool players are said to have 'imperfect aim'. It is not unreasonable to assume a Gaussian distribution of impact parameters. Thus, one can model both head-on and grazing, or 'slice', collisions between two billiard balls as two Gaussian functions. Both collision types will have identical widths but will be centered at small values of b for head-on collisions and large values of b for grazing collisions

$$P(b) \propto \exp\left[-\frac{(b-fd)^2)}{2\sigma^2}\right] , \qquad (2)$$

where σ is the familiar standard deviation, and the full width at half maximum (FWHM) is $2\sqrt{2\ln 2}\,\sigma$. Here the slice coefficient, $0 \le f \le 1$, represents the fraction of d at which the impact occurs. As an example, let us assume that the two $P(b)$ functions have $\sigma = 0.02d$ so that FWHM is approximately $0.047d$. The head-on collision is for $f = 0.05$, i.e., centered at $b = 0.05d$, whereas the 'cut' shot impact parameter distribution is for $f = 0.95$, centered at $b = 0.95d$. The two probability distribution functions are plotted in *Figure* 3a. The question then is as follows: What do the angular probability distribution functions, $P(\theta)$, look like in the COM frame?

Just like for any two probability distribution functions, $P(b)$ and $P(\theta)$ are related by:

$$P(\theta) = P\left(b(\theta)\right) \cdot \left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| . \qquad (3)$$

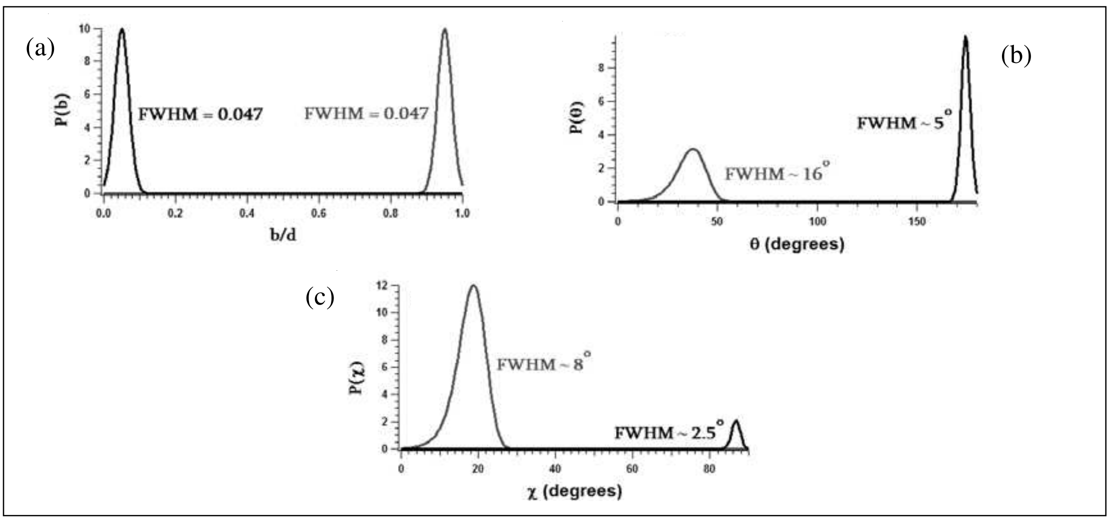
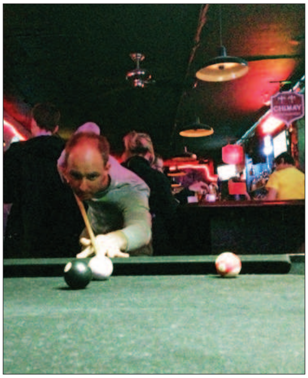


Figure 3. (a) Gaussian impact parameter distribution functions, $P(b)$, for head-on ($b = 0.05d$, black curve) and grazing ($b = 0.95d$, red curve) collisions. **(b)** The resulting angular distribution functions, $P(\theta)$, in the center-of-mass (COM) frame for head-on (black curve) and grazing collisions (red curve). **(c)** Angular distribution functions $P(\chi)$ in the laboratory (LAB) reference frame, i.e., as would be observed during an actual game. Note that even though the two impact parameter distribution functions have identical widths, the resulting angular distribution functions do not. This serves as a visual explanation of why grazing or 'slice' shots are more difficult than the head-on ones.


Justin Jankunas doing research for this article.

To compute $P(\theta)$ for a given $P(b)$ form, one needs the $\left|\frac{db}{d\theta}\right|$ term which is easily obtained from (1b). Thus, for $P(b)$ given by (2), the angular probability distribution function is given by:

$$P(\theta) \propto \exp\left[-\frac{(\cos(\theta/2) - f^2}{2\sigma'^2}\right] \cdot \sin\left(\frac{\theta}{2}\right) \,, \qquad (4)$$

where $\sigma' = \sigma/d$. The two $P(\theta)$ functions with $\sigma' = 0.02$ and $f = 0.05$ and $f = 0.95$ are plotted in *Figure* 3b.

The difference between *Figures* 3a and 3b is striking. Even though both Gaussian distributions have identical spreads in the impact parameter space – in other words, equally good aim for head-on and grazing shots – the spread in the angular space is not the same for head-on collisions ($\theta \sim 180^o$) and 'slice' shots ($\theta \sim 0^o$). The spread in the scattering angle θ is larger for 'cut' shots than for head-on shots! In our particular example, we have FWHM ($f= 0.05$) = 5^o and FWHM ($f= 0.95$) = 16^o.

This example explains why the so-called 'slice' or 'cut' shots are more difficult than the head-on shots in any game of billiards. The reasons are two-fold: the particular form of (1a), and the existence of a spread in the $P(b)$ distribution.

So far, we have analyzed the collision dynamics in the COM frame where $0^o \leq \theta \leq 180^o$, but the pool player views the shot in the frame of the pool hall, that is, the laboratory frame (LAB). It is well known that the largest angle through which an object ball can be 'cut' is $\sim 90^o$ in LAB, in which case the cue ball is barely deflected.Therefore, the scattering angle χ should run from 0^o to 90^o in the LAB frame. The two reference frames are related by [1]:

$$P(\chi) = 4\cos\left(\frac{\theta}{2}\right) P(\theta) \,, \qquad (5)$$

where $\chi = \theta/2$. $P(\chi)$ is plotted in *Figure* 3c. The main conclusions remain the same: The forward scattered cue ball, which is the 'cut' shot, has a larger uncertainty (spread) in the angular space than the head-on shot[2].

[2] Once again, care should be taken when defining scattering angles in the LAB frame. The angle χ plotted in *Figure* 3c refers to the angle through which the cue ball is "deflected. If one followed the object ball, the angular distribution "functions would remain the same, but the χ axis would flip though the 45° point in *Figure* 3c.

An astute reader might object at this point. A common statement about the hard sphere collisions that can be found in any scattering textbook is that, 'Hard sphere scattering is isotropic' [2]. Results obtained herein and plotted in *Figures* 3b and 3c are far from isotropic. What is going on?

Hard sphere scattering is indeed isotropic in three dimensions. The differential cross-section, $\mathrm{d}\sigma/\mathrm{d}\Omega$, is defined as a ratio of the number of particles, N, scattered into a particular solid angle, $\mathrm{d}\Omega$, over the number of incident particles, N_o, within a certain impact parameter range $\mathrm{d}b$, in other words $2\pi b\mathrm{d}bN_\mathrm{o} = 2\pi\sin\theta\mathrm{d}N$. Rearranging, and substituting the expressions for b and $\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right|$ yields

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \equiv \frac{N}{N_\mathrm{o}} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| = \frac{d\cos(\theta/2)}{\sin\theta}\frac{d\sin(\theta/2)}{2} = \frac{d^2}{4}\,. \tag{6}$$

Textbooks rightly claim the hard sphere scattering to be isotropic! In two dimensional collisions, as in a game of pool, one has $\mathrm{d}bN_\mathrm{o} = \mathrm{d}\theta N$, which yields

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \equiv \frac{N}{N_\mathrm{o}} = \left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| = \frac{d\sin(\theta/2)}{2}\,, \tag{7}$$

i.e., the differential cross-section is anisotropic!

Although hard-sphere scattering is of limited relevance in molecular collisions, mainly because atoms and molecules almost always exhibit finite degrees of attraction and repulsion, its use elsewhere can be fun and even pedagogical! This quick study explains what any billiards player has always intuitively known: 'slice' shots often bring disappointed sighs and headshakes, but situations in which a cue ball, an object ball and a pocket lie on a more or less straight line are often a desired 'leave' of any given shot and bring on smiles and fist pumps of triumph. This is primarily caused by 'imperfect aim' which basically is a finite spread in the impact parameter distribution function. To paraphrase what Marc Anthony said about Brutus in Shakespeare's *Julius Caesar,* slice shots "are the most unkindest cut of all" [3].

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Suggested Reading

[1] N E Henriksen and F Y Hansen, *Theories of Molecular Reaction Dynamics*: *The Microscopic Foundation of Chemical Kinetics*, Oxford University Press, Oxford, 2008.

[2] R D Levine and R B Bernstein, *Molecular Reaction Dynamics and Chemical Reactivity*, Oxford University Press, New York, p.73, 1987.

[3] W Shakespeare, *Julius Caesar,* Act 3, Scene 2, pp.181–186.