Statistical mechanics deals with a collection of a large number of identical non-interacting particles, like the molecules of an ideal gas. Each particle can exist in different energy states. Given the total number of particles and the total energy of the collection of particles, statistical mechanics tries to find the most probable distribution of the particles in the different energy states.

In classical statistics of an ideal gas, the individual molecules can have all values of kinetic energy from zero to infinity, and the most probable distribution of the kinetic energy is given by the Maxwellian distribution. Such a distribution leads to the result that the specific heat at constant volume of each molecule should be $3k/2$, independent of temperature. This result is in contradiction with the Third Law of Thermodynamics (the so-called Nernst Heat Theorem), which requires the specific heat of all materials to tend to zero as the absolute zero of temperature is approached.

After Planck postulated his quantum hypothesis to explain the black-body radiation spectrum, it became apparent that quantization of a system is required to explain results which depart significantly from predictions of classical physics. The specific heat of an ideal gas molecule is one such situation.

When a system is quantized, the energy levels become discrete, and the classical assumption that the energy can vary continuously no longer holds. In the case of an ideal gas molecule confined to a certain volume, the separation of the energy levels will depend on the volume. In the limit the volume tends to infinity, the discrete character of the spectrum will approach the continuous distribution postulated by classical physics. So quantization alone will not resolve the specific heat conundrum.

In 1925 Pauli postulated that a given quantum state of an electron in an atom, characterized by a complete set of quantum numbers, can be occupied by only one electron. This postulate was eminently successful in explaining the structure of the atom and its position in the periodic table of elements, as well as several spectroscopic phenomena. Such a postulate implies that any quantum state with a given energy can be occupied by only one electron. This has serious implications in the distribution of the particles in different quantum states, and, therefore, in finding the most probable distribution of a given number of particles with a given total energy.

Fermi realized the significance of the Pauli principle in working out the statistics of a system of particles obeying that principle. He applied it to the ideal gas, assuming that the molecules obey the Pauli principle. He arrived at the important conclusion that there is a zero-point energy and zero-point pressure in such an ideal gas at absolute zero of temperature, and the specific heat of such a gas will be proportional to the absolute temperature, as one approaches absolute zero. Nowadays
one calls this statistics the Fermi–Dirac statistics and one applies it to the free electron gas in a metal.

However, Fermi was wrong in assuming the Pauli principle to be valid for all gas molecules. In his paper, the translation of which is presented below, he has stated that the pressure in a gas of non-interacting helium atoms will deviate by 15% from the result of classical statistics at 5K and 10 atmospheres pressure. The connection between spin and statistics came later. We now know that atoms which have a total spin quantum number (including nuclear spin) that is an odd half integral multiple of $\hbar/2\pi$, obey Fermi–Dirac statistics, while those atoms which have a total spin quantum number that is zero or an integral multiple of $\hbar/2\pi$, obey a different quantum statistics called Bose–Einstein statistics. The commonly occurring isotope of helium is $^4$He which obeys Bose–Einstein statistics and not Fermi–Dirac statistics. So Fermi’s calculation of zero-point pressure for helium gas is erroneous. However Fermi’s result will apply to the rare isotope of Helium, $^3$He, which obeys Fermi statistics.

At low temperatures and pressures, which are normally used, the interaction between the atoms will produce deviations from the ideal gas laws which will swamp the predictions of quantum statistics for an ideal non-interacting gas.

It is now possible, using laser cooling, to produce a cloud of atoms at a pressure of $10^{-10}$ mbar or less and at temperatures of a few nano Kelvin so that quantum degeneracy effects can be studied. The low density of the gas reduces the effects of interaction between the atoms to insignificant levels. Studies have been made both on atoms obeying Bose–Einstein statistics and Fermi–Dirac statistics to measure quantum degeneracy effects and compare them with the theory of non-interacting molecules.

I consider it a privilege to have been given the opportunity to translate Fermi’s epoch-making paper and to write an introduction to the readers of Resonance.

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R Srinivasan is an eminent physicist who has worked at IIT (Chennai), DAE Facility at Indore and at RRI (Bangalore). In recent years, he is fully engaged in improving the standard of teaching experimental physics in the country. For this purpose he has developed some relatively newer techniques of measurement made possible by the availability of inexpensive analogue chips. Since the year 2000 he has conducted 54 two-week Refresher Courses all over the country, currently at the rate of about 10 per year, training over 1200 teachers and several research students in physics. He has set up a permanent laboratory for conducting courses in experimental physics and materials science at the premises of the Indian Academy of Sciences in Bangalore.
If the Nernst Heat Theorem should retain its validity even for the ideal gas, one must assume that the ideal gas law should deviate from the classical law at low temperatures. The cause for this deviation is to be sought in the quantization of molecular motions. In all theories to explain the deviation, more or less arbitrary assumptions have been made on the statistical behavior of molecules, or on their quantization. In the present work the only assumption used is that, in a system, no two equivalent molecules can exist with all quantum numbers identical. This assumption was first proposed by Pauli and is based on numerous spectroscopic facts. With this assumption, the equation of state and the internal energy of an ideal gas is derived; the entropy value at high temperatures agrees with the Stern Tetrode relation.

In classical thermodynamics the molecular heat (at constant volume) (of an ideal gas molecule) is fixed at

\[ c = \frac{3k}{2T^*} \, . \quad (1) \]

But if one likes to use the Nernst Heat Theorem also for an ideal gas, one must consider equation (1) to be merely an approximation to the behavior at high temperatures, since \( c \) must vanish in the limit \( T \to 0 \). It is therefore necessary to accept that the motion of a molecule of an ideal gas should be quantized; this quantization manifests itself in certain degeneracy phenomena at low temperatures, so that the specific heat, as well as the equation of state, deviate from their classical expressions.

The purpose of the work below is to establish a method for quantization of an ideal gas, which, in our opinion, is, as much as possible, independent of arbitrary assumptions on the statistical behavior of the gas molecule. In recent times many attempts have been made to determine the equation of

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1 See the provisional communication, Lincei Rend (6),3,145, 1926.

* Please note: the correct equation (1) is \( c = \frac{3k}{2} \).
state of an ideal gas\(^2\). The equations of state of different authors and the one proposed by us differ from one another, and from the classical equation of state \(pV = NkT\), through terms, which become considerable only at low temperatures and high pressures. Unfortunately the deviations of a real gas from an ideal gas are also largest under these conditions, so that degeneracy effects could not be observed unambiguously so far. However it may not be impossible that a deeper knowledge of the equation of state may allow us to separate effects of degeneracy from the remaining deviations from the equation \(pV = NkT\), and may enable experiment to decide between the different degeneracy theories.

In order to apply the quantization rules to the motion of a molecule of our ideal gas, one can proceed in different ways; but the end result is always the same. For example, we may assume the molecule to be enclosed in a vessel in the shape of a parallelopiped with reflecting walls. Then the motion of the flying molecule back and forth between the two walls is forced to be periodic and can, therefore, be quantized. More generally one can think that the molecule is placed in such an external force field that its motion is forced to be periodic. The assumption that the gas is ideal enables us to neglect the mutual interactions of the molecules so that the mechanical motion is solely determined by the external force. However, it is apparent that quantization of molecular motion on the assumption of complete independence of molecules from one another, is not sufficient to account for the expected deviation (of the specific heat). This is recognized best in the example of a molecule enclosed in a vessel; when the linear dimensions of the vessel are increased the energy values of the quantum states of each molecule come closer and closer together so that, for a vessel of macroscopic dimensions, the effect of the discontinuity in the energy values practically vanishes. Besides, this effect depends on the volume of the vessel even when the number of molecules contained in the vessel is so chosen that the density remains constant.

\(^2\) See for example A. Einstein, Berl. Ber. 1924, page 261; 1925, page 318.; M Planck. Ibid 1925, page 49. Our method as well as that of Einstein abandons the idea of statistical independence of the molecules. However the type of dependence in our work differs completely from that used by Einstein. The end result for the deviation from the classical equation of state in our work is found to be opposite to that in the work of Einstein.
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Through a quantitative calculation of this situation\(^3\) it can be shown that a deviation of the expected magnitude occurs only when the vessel is chosen so small that on an average it contains only one molecule.

We therefore presume that an additional rule on the Sommerfeld Quantum conditions is required when quantizing an ideal gas.

Recently, Pauli\(^4\), in the context of the work of E. C. Stoner\(^5\), postulated a rule that, when an electron exists in an atom in an orbit with specified quantum numbers (including the magnetic quantum number), no other electron can occupy the quantum orbit characterized by the same numbers. In other words a quantum orbit (in an external magnetic field) can be occupied only by one electron.

Since this rule of Pauli has been shown to be extremely fruitful in explaining spectroscopic facts\(^6\), we wished to test whether it can be used even for the problem of quantization of the ideal gas.

We will show that this indeed is the ease and the application of Pauli’s rule enables us to present a complete consistent theory for the degeneracy of an ideal gas. In the following we will assume that, in a state with given quantum numbers, there can be at most one molecule in our ideal gas. We not only include the quantum numbers describing the internal motions of a molecule but also those describing its translational motion.

To begin with, we place our molecule in a suitable external field so that its motion is forced to be periodic. This can be achieved in an innumerable number of ways. Since the result does not depend on the choice of the force field, we subject the molecules to a central elastic force of attraction to a fixed point \(O\) (origin of coordinates), so that each molecule forms a harmonic oscillator. This central force will confine the gas to a region around \(O\). The gas density will decrease as the distance from \(O\) increases and will vanish at infinite distance. If \(\nu\) is the eigen-frequency of the oscillator, then the force acting on a molecule is given by

\[
4\pi^2 \nu^2 mr,
\]

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\(^3\) E. Fermi, Nuovo Cim. 1, 145, 1924.

\(^4\) W. Pauli jr. ZS. F. Phys. 31, 765, 1925

\(^5\) E.C. Stoner, Phil. Mag. 48, 719, 1924.

\(^6\) See for example F. Hund, ZS.f. Phys. 33, 315, 1925.
where \( m \) is the mass of the molecule and \( r \) its distance from \( O \). The potential energy of the attractive force is then

\[
  u = -2\pi^2\nu^2 mr^2. \tag{1}
\]

Let the quantum numbers of a molecular oscillator be \( s_1, s_2, \) and \( s_3 \). These quantum numbers are insufficient to characterize the state of the molecule. To these must be added the quantum numbers for internal motions. We will confine ourselves only to monatomic molecules and, further we will assume that all the molecules present in our gas are in the ground state and this state is single (i.e., it cannot be split by the application of a magnetic field). Then we do not have to bother with the internal motions, and can treat the molecule simply as a point mass. In our case the Pauli rule will then be the following: In the entire amount of gas, there can be at most one molecule in a state with the given quantum numbers \( s_1, s_2 \) and \( s_3 \).

The total energy of this molecule is given by

\[
  w = h\nu(s_1 + s_2 + s_3) = h\nu s. \tag{2}
\]

The total energy will therefore be an arbitrary integral multiple of \( h \). The value \( sh\nu \) can be realized in many ways. Every possibility of realization corresponds to a solution of the equation

\[
  s = s_1 + s_2 + s_3. \tag{3}
\]

where \( s_1, s_2 \) and \( s_3 \) can assume values \( 0, 1, 2, 3, \ldots \). Equation (3) has

\[
  Q_s = \frac{(s + 1)(s + 2)}{2} \tag{4}
\]
solutions. The energy \( 0 \) can only be realized in one way, the energy \( h\nu \) in three ways, the energy \( 2h\nu \) in six ways and so on. A molecule with energy \( sh\nu \) will be termed an \( s \)-molecule for simplicity.

According to our assumptions, in the entire gas, there can be at most \( Q_s \) \( s \)-molecules; there can be one molecule with zero energy, three molecules with energy \( h\nu \), six with energy \( 2h\nu \) \ldots and so on.

In order to see clearly the consequence of the above fact, we will consider the extreme case when our gas is at absolute zero. Let \( N \) be the number of
molecules. At absolute zero the gas must be in the state of lowest energy. If there was no constraint on the number of molecules with a given energy, then every molecule would be in the state of zero energy \( s_1 = s_2 = s_3 = 0 \). But according to what was said before, a maximum of one molecule can exist in a state of zero energy. If \( N = 1 \), this single molecule will occupy the zero energy state at absolute zero. If \( N = 4 \), one molecule will occupy the zero energy state, and the remaining molecules the three degenerate states with energy \( h\nu \). If \( N = 10 \), one molecule will be in the zero energy state, three molecules will occupy the three available states with energy \( h\nu \), and the remaining six molecules will occupy the energy states \( 2h\nu \) and so on.

At absolute zero the molecules of our gas will exhibit a shell-type structure, which shows a certain analogy with the shell-type arrangement of electrons in a multi-electron atom.

We will now investigate how a certain quantity of energy

\[
W = Eh\nu
\]

\((E\) is an integer\) will be distributed among the \( N \) molecules.

Let \( N_s \) be the number of molecules which are found in a state with energy \( sh\nu \). According to our assumption

\[
N_s = Q_s.
\]

We have also the equations

\[
\Sigma N_s = N,
\]

\[
\Sigma sN_s = E,
\]

which express the facts that the total number of molecules is \( N \) and the total energy is \( Eh\nu \).

We will find the number \( P \) of such arrangements that \( N_0 \) molecules are in state zero, \( N_1 \) are in state \( h\nu \), \( N_2 \) are in state \( 2h\nu \) and so on. Two arrangements in which the states occupied by the molecules are the same will be considered identical. In other words, two distributions which only differ by a permutation of the molecules among the states occupied will be considered identical (will be counted only once). If one were to consider two such arrangements as different, one must multiply \( P \) with the constant.
Now these equations are represented as:

\[
\binom{Q_s}{N_s}^*, \quad \binom{Q_0}{N_0} \binom{Q_1}{N_1} \binom{Q_2}{N_2} \ldots - \prod \binom{Q_s}{N_s}^*.
\]

Therefore for \( P \) we get the expression

\[
P = \binom{Q_0}{N_0} \binom{Q_1}{N_1} \binom{Q_2}{N_2} \ldots - \prod \binom{Q_s}{N_s}^*.
\]

One obtains the probable values of \( N_s \) by seeking the maximum of \( P \) subject to the constraints (7) and (8). Using the Stirling theorem we may write to a sufficient degree of approximation

\[
\log P = \sum \log \binom{Q_s}{N_s} = \sum \left( N_s \log \frac{N_s}{Q_s - N_s} - N_s \log \frac{Q_s - N_s}{Q_s} \right).
\]

We seek values of \( N_s \), that will satisfy equations (7) and (8), and make \( \log P \) a maximum. One finds

\[
\alpha e^{-\beta s} = \frac{N_s}{Q_s - N_s},
\]

where \( \alpha \) and \( \beta \) are constants. The above equation gives

\[
N_s = Q_s - \frac{\alpha e^{-\beta s}}{1 + \alpha e^{-\beta s}}.
\]

The values of \( \alpha \) and \( \beta \) are determined by equations (7) and (8), or conversely one may take \( \alpha \) and \( \beta \) to be given and determine the total number of particles and the total energy through equations (7) and (8).  

\[\text{QsC}_{N_s}\]

\[
P = Q_0 C_{N_0} Q_1 C_{N_1} Q_2 C_{N_2} = \prod Q_s C_{N_s}
\]
We find

\[ N = \sum_{0}^{\infty} Q_s \frac{\alpha e^{-\beta s}}{1 + \alpha e^{-\beta s}} \]

\[ \frac{W}{h \nu} - E = \sum_{0}^{\infty} Q_s \frac{\alpha e^{-\beta s}}{1 + \alpha e^{-\beta s}} \]

The absolute temperature \( T \) of the gas is a function of \( N \) and \( E \) or it is also a function of \( \alpha \) and \( \beta \). This function can be determined by two methods, which nevertheless lead to the same result. For example one can set, according to the Boltzmann Principle of Entropy,

\[ S = k \log P \]

and calculate the temperature from the formula

\[ T = \frac{dW}{dS} \]

However, like all methods based on Boltzmann’s principle, this method has the disadvantage that for its use one must make a more or less arbitrary hypothesis for the state probability. We therefore prefer to proceed as follows. We note the density of our gas is a function of the distance from the center and it vanishes for infinite distance. For infinite \( r \), the degeneracy phenomena are lost, and the statistics of our gas must be the classical statistics. Especially for \( r \to \infty \), the average kinetic energy of the molecule must be \( 3kT/2 \), and the velocity distribution must be Maxwellian. We can therefore determine the temperature from the velocity distribution in the regime of infinitesimally small density. Since the entire quantity of gas must be at the same temperature, we then know the temperature even in regions of high density. It is like saying that we use a gas thermometer with an infinitely dilute gas for the determination of temperature.

First we have to calculate the density of molecules with a kinetic energy between \( L \) and \( L + dL \) at a distance \( r \). The total energy of this molecule will be between

\[ L + 2\pi^2 \nu^2 m r^2 \quad \text{and} \quad L + 2\pi^2 \nu^2 m r^2 + dL \]
from (1). The total energy of a molecule is \( shv \). For our molecule \( s \) must therefore lie between \( s \) and \( s + ds \), where

\[
\frac{s}{L} = \frac{2\pi^2 \nu m r^2}{h}, \quad ds = \frac{dL}{h}. \tag{13}
\]

We will consider for the present one molecule whose motion is characterized by the quantum numbers \( s_1, s_2, \) and \( s_3 \). Its coordinates \( x, y, z \) are given as functions of time through

\[
\begin{align*}
   x - \sqrt{Hs_1} \cos(2\pi vt - \alpha_1), & \quad y - \sqrt{Hs_2} \cos(2\pi vt - \alpha_2), \\
   z - \sqrt{Hs_3} \cos(2\pi vt - \alpha_3).
\end{align*} \tag{14}
\]

Here

\[
\mu = \frac{h}{2\pi^2 \nu m} \tag{15}
\]

and \( \alpha_1, \alpha_2, \alpha_3 \) are phase constants, which can take an arbitrary set of values with equal probability. From (14) it follows that \( |x| \leq \sqrt{Hs_1}, |y| \leq \sqrt{Hs_2} \) and \( |z| \leq \sqrt{Hs_3} \). The probability that \( x, y \) and \( z \) lie between the limits \( x \) and \( x + dx \), \( y \) and \( y + dy \) and \( z \) and \( z + dz \) is given by

\[
\frac{dxdydz}{\pi^3 \sqrt{(Hs_1 - x^2)(Hs_2 - y^2)(Hs_3 - z^2)}}.
\]

If we do not know the individual values of \( s_1, s_2 \) and \( s_3 \), but only know their sum \( s \), then the probability is given by

\[
\frac{1}{Q_s} \frac{dxdydz}{\pi^3 \sum \sqrt{(Hs_1 - x^2)(Hs_2 - y^2)(Hs_3 - z^2)}}. \tag{16}
\]

The sum is taken over all integer solutions of equation (3) satisfying the conditions

\[
Hs_1 \geq x^2, \ Hs_2 \geq y^2, \ Hs_3 \geq z^2.
\]

When we multiply the probability (16) by the number \( N_s \) of \( s \)-molecules, we obtain the number of \( s \)-molecules in the volume \( dxdydz \). Taking into account equation (11), we then find the density of \( s \)-molecules at position \( x, y, z \) to be given by
\[ n_s = \frac{\alpha e^{-\beta s}}{1 + \alpha e^{-\beta s}} \frac{1}{\pi^3} \sum \frac{1}{\sqrt{(H s_1 - x^2)(H s_2 - y^2)(H s_3 - z^2)}}. \]

For sufficiently large \( s \) the sum can be replaced by a double integral. After carrying out the integration we get

\[ n_s = \frac{2}{\pi^2 H^2} \frac{\alpha e^{-\beta s}}{1 + \alpha e^{-\beta s}} \sqrt{(H s - r^2)}. \]

Using (13) and (15) we find that the density of molecules with a kinetic energy between \( L \) and \( L + dL \) at the point \( x, y, z \) is expressed by the following

\[ n(L) dL = n_s dL - \frac{2\pi}{h^3} \sqrt{L} dL \frac{\alpha e^{-2\pi^2 \nu_0^2 s^2}}{1 + \nu_0} \frac{\beta L}{h^3} e^{\frac{\beta L}{h^3}}. \quad (17) \]

This formula must be compared with the classical Maxwellian distribution law

\[ n^*(L) dL = K \sqrt{L} dL e^{-L/kT} \quad (17') \]

We then see that in the limit \( r \to \infty \) (17) goes over to (17') only when

\[ \beta = \frac{h\nu}{kT}. \quad (18) \]

Now (17) can be rewritten as

\[ n(L) dL = \frac{(2\pi)(2m)^{3/2}}{h^3} \sqrt{L} dL \frac{A e^{-L/kT}}{1 + A e^{-L/kT}} \quad (19) \]

where

\[ A = \frac{\alpha e^{-2\pi^2 \nu_0^2 mT^2}}{kT}. \quad (20) \]

The total number of molecules at a distance \( r \) is

\[ n = \int_0^{r_0} n(L) dL = \frac{(2\pi m k T)^{3/2}}{h^3} F(A), \quad (21) \]
where
\[ F(A) = \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{A\sqrt{x}e^{-x}}{1 + Ae^{-x}} \, dx, \quad (22) \]

The average kinetic energy of a molecule at a distance \( r \) is
\[ L = \frac{1}{n} \sum_{i=1}^{\infty} \ln(L_i) \, dL_i = \frac{3 kT}{2} \frac{G(A)}{F(A)}, \quad (23) \]

where
\[ G(A) = 4 \frac{4}{3\sqrt{\pi}} \int_0^\infty \frac{A x^{3/2}e^{-x}}{1 + Ae^{-x}} \, dx. \quad (24) \]

We can determine \( A \) as a function of density and temperature through (21). When the value of \( A \) so found is inserted in (19) and (23) we get the velocity distribution and the average kinetic energy as a function of density and temperature.

To arrive at the equation of state we use the Virial theorem. According to this theorem the pressure \( p \) is given by
\[ p = \frac{2}{3} n \bar{L} = n kT \frac{G(A)}{F(A)}. \quad (25) \]

The value of \( A \) is taken as a function of density and temperature through equation (21).

Before we proceed further, we state a few mathematical properties of the functions \( F(A) \) and \( G(A) \) introduced above.

For \( A \leq 1 \), both functions can be represented by the converging series
\[
\begin{align*}
F(A) &= A - \frac{A^2}{2^{3/2}} + \frac{A^3}{3^{3/2}} - \cdots, \\
G(A) &= A - \frac{A^2}{2^{5/2}} + \frac{A^3}{3^{5/2}} - \cdots
\end{align*}
\quad (26)
\]
For large \( A \) one has the following asymptotic expressions

\[
F(A) = \frac{4}{3\sqrt{\pi}} (\log A)^{3/2} \left[ 1 + \frac{\pi^2}{8(\log A)^2} + \cdots \right],
\]
\[
G(A) = \frac{8}{15\sqrt{\pi}} (\log A)^{5/2} \left[ 1 + \frac{5\pi^2}{8(\log A)^2} + \cdots \right].
\]  

(27)

Further the relation

\[
\frac{dG(A)}{F(A)} = d\log A.
\]  

(28)

also holds. We must introduce yet another function \( P(\Theta) \) defined by

\[
P(\Theta) = \Theta \frac{G(A)}{F(A)}, \quad F(A) = \frac{1}{\Theta^{3/2}}
\]  

(29)

For very large and very small values of \( \Theta \), the function \( P(\Theta) \) can be calculated respectively from the approximation formulae

\[
P(\Theta) = \Theta \left\{ 1 \mid \frac{1}{2^{3/2}\Theta^{3/2}} \mid \cdots \right\}
\]
\[
P(\Theta) = 3^{2/3}\Theta^{1/3} \left\{ 1 \mid 5 \times 2^{2/3}\Theta^{4/3} \mid 3^{7/3} \Theta^2 \mid \cdots \right\}.
\]  

(30)

Using (29), (28), (27) we see further that

\[
\int_0^\Theta \frac{dP(\Theta)}{\Theta} = \frac{5}{3} \frac{G(A)}{F(A)} = \frac{2}{3} \log A.
\]  

(31)

We are now in a position to eliminate the parameter \( A \) from the Equation of State (25) and the equation (23), and we find the pressure \( p \) and the mean kinetic energy of the molecule as explicit functions of density and temperature:

\[
p = \frac{h^2n^{5/3}}{2\pi m} P\left(\frac{2\pi m kT}{h^2n^{2/3}}\right),
\]  

(32)
\[ L = \frac{3\hbar^2 n^{2/3}}{2\pi m} \left( \frac{2\pi mkT}{\hbar^2 n^{2/3}} \right)^{2/3} \]  
(33)

In the limiting case of weak degeneracy (\( T \) is large and \( n \) is small) the equation of state takes the form
\[ p - nkT \left\{ 1 + \frac{1}{16} \frac{\hbar^3 n}{(\pi mkT)^{3/2}} + \ldots \right\} \]  
(34)

Therefore, the pressure is more than that given by the classical equation of state \( p - nkT \). For an ideal gas with the atomic weight of helium, at \( T = 5^\circ \) and a pressure of 10 atmospheres, the difference amounts approximately to 15%.

In the limit of large degeneracy (32) and (33) assume the form
\[ p = \frac{1}{20} \left( \frac{6}{\pi} \right)^{2/3} \frac{h^2 n^{5/3}}{m} + \frac{2^{4/3} \pi^{8/3}}{3^{2/3}} \frac{m n^{1/3} k^2 T^2}{\hbar^2} + \ldots \]  
(35)

\[ \bar{L} = \frac{3}{40} \left( \frac{6}{\pi} \right)^{3/2} \frac{h^2 n^{2/3}}{m} + \frac{2^{1/3} \pi^{8/3}}{3^{2/3}} \frac{m k^2 T^2}{\hbar^2 n^{2/3}} + \ldots \]  
(36)

One recognizes that degeneracy results in a zero-point pressure and a zero-point energy. From (36) the specific heat at low temperature can be calculated. One finds
\[ c_v = \frac{d\bar{L}}{dT} = \frac{2^{4/3} \pi^{8/3}}{3^{2/3}} \frac{mk^2 T}{\hbar^2 n^{2/3}} + \ldots \]  
(37)

We see that the specific heat vanishes at absolute zero and is indeed proportional to the absolute temperature at low temperatures. Finally, we will show that our theory leads to the Stern–Tetrode value for the absolute entropy of gases. Using (33) one finds actually
\[ S = n \int_0^T \frac{d\bar{L}}{T} - \frac{3}{2} nk \int_0^\Theta \frac{P(\Theta)}{\Theta} d\Theta. \]

(31) then gives
\[ S = nk \left\{ \frac{5}{2} \frac{G(A)}{F(A)} - \log A \right\}, \]  
(38)
where the value of $A$ is to be taken from (21). For high temperatures, we find using (26)

$$A = \frac{\hbar^3}{(2\pi m k T)^{3/2}}, \quad \frac{G(A)}{F(A)} = 1.$$ 

Equation (38) then gives

$$S = n k \left\{ \log \frac{(2\pi m k T)^{3/2}}{\hbar^3} + \frac{5}{2} \right\}$$

$$- n k \left\{ \frac{3}{2} \log T - \log n + \log \frac{(2\pi m)^{3/2} k^{3/2} e^{3/2}}{\hbar^3} \right\},$$

This agrees with the entropy value of Stern and Tetrode.