Littlewood and Number Theory

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J E Littlewood (1885–1977) was a British mathematician well known for his joint work with G H Hardy on Waring's problem and the development of the circle method. In the first quarter of the 20th century, they created a school of analysis considered the best in the world. Littlewood firmly believed that research should be offset by a certain amount of teaching. In this exposition, we will highlight several notable results obtained by Littlewood in the area of number theory.

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1. Introduction

John Edensor Littlewood was one of the most well-known mathematicians during the first part of the 20th century. Most notable among his doctoral students were A O L Atkin, Sarvadaman Chowla, Harold Davenport, A E Ingham and H P F Swinnerton-Dyer. For official reasons, Srinivasa Ramanujan is also considered as one of Littlewood's students.

In his obituary on Littlewood, Burkill [1] observes that around 1900, there was no British school of analysis worthy of the name and it is due to G H Hardy and J E Littlewood that England owes a debt of gratitude for building by 1930 "a school of analysis second to none in the world."

2. Prime Numbers

Littlewood's first result in prime number theory concerned the error term in the prime counting function. If $\pi(x)$ denotes the number of primes up to x, then the prime number theorem tells us that

$$\pi(x) \sim \operatorname{li} x,$$

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Littlewood proved the surprising fact that the difference $\pi(x) - \mathrm{li}(x)$ between the prime counting function $\pi(x)$ and the logarithmic integral $\mathrm{li}(x)$ changes sign infinitely often.

where

$$\mathrm{li}\, x = \int_2^x \frac{\mathrm{d}t}{\log t}$$

is called the logarithmic integral. Here, the symbol $A(x) \sim B(x)$ means that $A(x)/B(x)$ tends to 1 as x tends to infinity. Numerical evidence seemed to suggest that

$$\pi(x) < \mathrm{li}\, x,$$

but Littlewood showed that this was incorrect. In an important and substantial paper, he [2] showed that

$$\pi(x) - \mathrm{li}\, x$$

changes sign infinitely often. More precisely, he proved that there are positive constants c_1, c_2 such that

$$\pi(x) - \mathrm{li}\, x > c_1 \frac{x^{1/2} \log\log\log x}{\log x}$$

for infinitely many x tending to infinity, and

$$\pi(x) - \mathrm{li}\, x < -c_2 \frac{x^{1/2} \log\log\log x}{\log x}$$

for infinitely many x tending to infinity. One usually writes this fact as

$$\pi(x) - \mathrm{li}\, x = \Omega_{\pm} \left(\frac{x^{1/2} \log\log\log x}{\log x} \right).$$

3. The Approximate Functional Equation of the Zeta Function

However, what distinguishes Littlewood in the history of mathematics is his prolific and productive collaboration with G H Hardy. Together, they tackled some of the fundamental questions regarding the Riemann zeta function and related questions. As is well known, the Riemann zeta function is defined by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \Re(s) > 1.$$

By virtue of the unique factorization of the natural numbers one can write this as an infinite product over prime numbers:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

for $\Re(s) > 1$. In his 1860 paper, Riemann derived the analytic continuation and functional equation for $\zeta(s)$. More precisely, he showed that

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Here, the Euler gamma function is defined by $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\mathrm{d}x$ for $\Re(z) > 0$. Beautiful as the above functional equation is, it is impractical to use in many questions of analytic number theory. In 1921, Hardy and Littlewood [3] derived the following approximate functional equation:

$$\zeta(s) = \sum_{n\le x} n^{-s} + \chi(s)\sum_{n\le y} n^{-(1-s)} + O(x^{-\sigma}) + O(|t|^{1/2-\sigma}y^{\sigma-1}),$$

where $s = \sigma + it$, t real, and $2\pi xy = |t|$, with

$$\chi(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\frac{\pi s}{2}.$$

Here, we write for non-negative $b(x)$, $a(x) = O(b(x))$ if there is a constant K such that $|a(x)| \le Kb(x)$ for all x. Using this, they were able to show that the number of zeros of $\zeta(s)$ with $|\Im(s)| \le T$ that lie on the line $\Re(s) = 1/2$ is greater than cT for some positive constant c. (Of course, the celebrated Riemann hypothesis says that *all* of the non-trivial zeros of $\zeta(s)$ lie on the line $\Re(s) = 1/2$.) The Hardy–Littlewood theorem was improved by Selberg [4] in 1941 who showed that a positive proportion of the non-trivial zeros of $\zeta(s)$ lie on the critical line. In 1974, Levinson [5] showed that one-third

Hardy and Littlewood showed that the Riemann zeta function has infinitely many zeroes on the critical line $\Re(s) = 1/2$.

of the zeros lie on the critical line and in 1989, Conrey [6] improved this to show that two-fifths of the zeros lie on the critical line. The seminal work of Hardy and Littlewood initiated these subsequent developments.

Another important application of the approximate functional equation is the beautiful asymptotic formula:

$$\int_0^T |\zeta(\frac{1}{2}+it)|^2 \mathrm{d}t \sim T \log T.$$

In 1934, E.C. Titchmarsh [7] improved this to show

$$\int_0^T |\zeta(\frac{1}{2}+it)|^2 \mathrm{d}t = T(\log T - \log 2\pi + 2\gamma - 1) + O(T^{\theta+\epsilon}),$$

where γ is Euler's constant and $\theta = 5/12$. In his TIFR doctoral thesis, R Balasubramanian [8] improved this further and showed that one can take $\theta = 1/3$.

4. Littlewood's Problem on Diophantine Approximation

For any real number x, let $||x||$ denote the distance of x to the nearest integer. For any pair of real numbers α, β, Littlewood conjectured that

The possible exceptions to Littlewood's conjecture from around 1930 on simultaneous approximation of two real numbers by rationals with the same denominator, have been proved in 2006 to have zero Hausdorff dimension.

$$\liminf_{n\to\infty} n||n\alpha||||n\beta|| = 0.$$

Another way to formulate this is that for any $\epsilon > 0$, the inequality

$$|x(x\alpha - y)(x\beta - z)| < \epsilon,$$

can be solved with $x \neq 0$ and x, y, z integers. The function $L(x, y, z) = x(x\alpha - y)(x\beta - z)$ is a product of three linear forms which admits a two-dimensional torus as a group of automorphisms. Recently, this conjecture has received considerable attention because it fits into a larger framework of ergodic theory and Lie theory. Using new methods from these theories, the authors in [9] showed that the set of exceptions to Littlewood's conjecture has Hausdorff dimension zero.

5. Waring's Problem and the Development of the Circle Method

The circle method originates with Ramanujan as can be seen by entries of his first letter to G H Hardy in 1913 (see for example, (1.14) on page 8 of [10] as well as Selberg [11]). After his arrival in England, Hardy and Ramanujan applied the method to derive the asymptotic formula for the partition function. After Ramanujan's premature demise, Hardy and Littlewood developed this method further and applied it first to tackle Waring's problem which we now explain.

A celebrated theorem of Lagrange states that every natural number can be written as the sum of four square numbers. Using elementary number theory, one can show that numbers of the form $8n+7$ cannot be written as the sum of three or fewer squares. Indeed, any square mod 8 is congruent to 0, 1 or 4 and it is impossible to choose three of these numbers (with replacement) to obtain 7 (mod 8). In 1770, Edmund Waring asked if generally, for any value of k, there is a number $g(k)$ such that every number can be written as a sum of $g(k)$ number of k-th powers and such that nothing smaller than $g(k)$ works. Thus, Lagrange's theorem states $g(2) = 4$. Waring conjectured further that $g(3) = 9$, $g(4) = 19$ and so on. That $g(k)$ exists was shown by Hilbert in 1909. However, obtaining an explicit formula for $g(k)$ was another matter. Hilbert's method, though elementary in the technical sense of the word, employed complicated algebraic identities and it was difficult to obtain any further information regarding $g(k)$ from his method. Moreover, if n can be written as a sum of r k-th powers, is it possible to obtain an asymptotic formula for the number of such representations? The answers to these questions were not accessible by Hilbert's method. They

The circle method is powerful enough to obtain asymptotic expressions for the number of representations of a positive integer as the sum of a (fixed) number of *k*-th powers.

were however obtainable via the powerful circle method as developed by Hardy and Littlewood in their series of papers on the subject (see for instance, [12]).

It is easy to see that to write the number 3^k-1 as a sum of k-th powers, we can only use 1 and 2^k. To use the minimum number of k-th powers, we use $([(3/2)^k]-1)$ 2^k's and the remaining 2^k-1 summands are all 1 so that

$$g(k) \geq 2^k + [(3/2)^k] - 2.$$

It seems that J A Euler, the son of the famous Leonard Euler, was the first to see this. Thus, $g(k) \geq 2^k + [(3/2)^k]-2$ for all values of k. In 1933, Pillai (see p. xiii of [13]) proved that $g(k) = 2^k + [(3/2)^k] - 2$ provided the following inequality holds:

$$(3/2)^k - [(3/2)^k] \leq 1 - (3/4)^k.$$

In 1957, Mahler [14] showed that the inequality is true for k sufficiently large, but was unable to determine how large, since his proof was ineffective. This is still an open problem today.

However, the powerful circle method enabled Hardy and Littlewood to obtain asymptotic formulas for $r_{k,s}(n)$, the number of ways of writing n as a sum of s k-th powers provided $s > 2^k$. More precisely, they showed that for $s > 2^k$,

$$r_{k,s}(n) \sim \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} S_{k,s}(n) n^{s/k-1},$$

where $S_{k,s}(n) \approx 1$. The factor $S_{k,s}(n)$ is called the *singular series* and can be written as a product over primes

$$S_{k,s}(n) = \prod_p A(p),$$

where $A(p)$ can be interpreted as a p-adic density in the following sense. For each natural number q, let $M(q)$ be the number of solutions of the congruence

$$x_1^k + \cdots + x_s^k \equiv n (\operatorname{mod} q).$$

Then,

$$A(p) = \lim_{r\to\infty} M(p^r)p^{-r(s-1)}.$$

Waring's problem is still an area of active research even today as is outlined in [15].

6. The Circle Method and Goldbach's Conjecture

After the successful series of papers on the Waring problem, Hardy and Littlewood realized that the method is applicable to study other questions such as the Goldbach's conjecture that every even number can be written as the sum of two primes and that every odd number can be written as the sum of three primes. Indeed, if $r_k(n)$ denotes the number of ways of writing n as a sum of k primes. Hardy and Littlewood [16] showed that for odd n, we have

$$r_3(n) \sim S_3(n)\frac{n^2}{2\log^3 n}, \qquad n\to\infty,$$

where

$$S_3(n) = \prod_p \left(1 + \frac{\delta_p(n)}{(p-1)^3}\right)$$

with $\delta_p(n) = -(p-1)$ if $p|n$ and 1 otherwise, *provided* the generalized Riemann hypothesis (GRH) is true. This hypothesis is the generalization of the classical Riemann hypothesis for the Riemann zeta function and pertains to Dirichlet L-series, $L(s,\chi)$ with χ a Dirichlet character. In 1937, Vinogradov derived some new estimates for exponential sums of the form

$$\sum_{p\le x} e^{2\pi i p\alpha},$$

that enabled him to remove the use of the GRH and so the Hardy–Littlewood theorem is now unconditional. For the binary Goldbach problem however, the method does not work. Hardy and Littlewood were able to

Although the circle method gives expressions for the number of ways of decomposing any large enough odd number as a sum of three primes, it is not sufficient to prove Goldbach's conjecture that every even number >2 is the sum of two primes.

use the method to conjecture an asymptotic formula for $r_2(n)$. Applying the same heuristic reasoning, they conjectured that the number of twin primes $p \leq x$ (that is primes p such that $p+2$ is also prime) is asymptotic to

$$S_2(2)\frac{x}{\log^2 x}$$

as x tends to infinity. In the same paper [16], they used the circle method to formulate a general conjecture about k-tuplets of primes. This conjecture says the following. Let $H = \{h_1, ..., h_k\}$ be a set of distinct natural numbers. For each prime p, let $\nu_p(H)$ be the size of the image of H (mod p). The Hardy–Littlewood conjecture is that the number of $n \leq x$ such that

$$n + h_1, ..., n + h_k$$

are all prime is asymptotic to

$$S(H)\frac{x}{\log^k x},$$

where $$S(H) = \prod_p \left(1 - \frac{\nu_p(H)}{p}\right)\left(1 - \frac{1}{p}\right)^{-k},$$

where the product is over primes p. Surprisingly, in the same paper they conjectured that

$$\pi(x+y) \leq \pi(x) + \pi(y),$$

for all $x, y \geq 2$. In 1974, Hensley and Richards [17] showed that this latter conjecture is incompatible with the k-tuplets conjecture and so one of them must be wrong. It is generally believed that it is the latter conjecture which is false, though no one has yet proved this is the case. The motivation for making the latter conjecture is based on the intuition that there are more primes in the 'initial' interval $[1, x]$ of length x than there are in the 'later' interval $[y, y+x]$ of length x. This example

Hardy and Littlewood conjectured that the prime counting function $\pi(x)$ satisfies $\pi(x+y) \leq \pi(x) + \pi(y)$ for all $x, y > 1$; it is expected to be false.

shows how sometimes, our intuition may be wrong, even for the best of minds.

7. Concluding Remarks

After the great collaborative work of Hardy and Ramanujan, the joint work of Hardy and Littlewood had acquired the status of epic collaboration in the annals of mathematics. Legend has it that they had four axioms for this collaborative work. The first was that when one wrote to the other it did not matter whether what was said was right or wrong, for otherwise, they could not write freely and openly as they pleased. The second axiom was that when one had received a message from the other, he was under no obligation to either read it or to respond to it. The third was that it did not matter if they both thought about the same detail. Finally, the fourth axiom was that it did not matter if one of them had contributed nothing to the paper under their joint names. These axioms apparently worked well for them and would probably work well for all collaborators. In his obituary of Littlewood, Burkill [1] comments that usually Littlewood would write the penultimate draft of their joint paper and Hardy would write the flowery prose of which he was a master.

Being at Trinity College in Cambridge, Littlewood enjoyed teaching. His remarks [18] regarding teaching and research seem very apropos today. He said, "I firmly believe that research should be offset by a certain amount of teaching, if only as a change from the agony of research. The trouble ... is that in practice you get either no teaching or else far too much."

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Of course, Littlewood is not only known for his collaborative work with Hardy, but his eminent contributions in other areas of analysis. It seems fitting to end this survey with a quotation of Littlewood that summarizes his philosophy: "Try a hard problem. You may not solve it, but you will prove something else."

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