Calculus and Geometry

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Sphere–Cylinder Theorem, volume and surface area of the torus, volume and surface area of a slice of a solid sphere.

In this article we compute volume and surface area of the torus, volume of a slice of a 3-ball. Besides this we discuss some interesting facts like 'derivative of the area of a circle with respect to its radius is its perimeter, derivative of the volume of the sphere with respect to its radius is its surface area and derivative of the volume of the torus with respect to the radius of its meridian circle is its surface area'.

1. Introduction

In school one learns how to compute area, volume, surface area and perimeter of various shapes like sphere, cone, cylinder and circle. But an equally important geometric object 'torus' – a shape like a scooter tube or a doughnut – is not discussed in school geometry. This is perhaps due to the non availability of this shape at the time when Archimedes (287 BC–212 BC) was computing volumes and surface areas of sphere, cone and cylinder. Had this shape been available at that time, Archimedes would have computed its volume and surface area and consequently today's school children might have been using these formulas along with the formulas of sphere, cone and cylinder. These days torus-like shapes are frequently seen at many places, so one would be interested to learn about their volumes and surface areas.

2. Derivatives of Areas and Volumes of Some Geometric Shapes

The area, $A(r) = \pi r^2$, of a circle of radius r is a polynomial function of a single variable r and its derivative with respect to r is $2\pi r$, which is its perimeter. The volume, $V(r) = \frac{4}{3}\pi r^3$, of a sphere of radius r is a also a polynomial function of a single variable r and its derivative $4\pi r^2$ is its surface area. The volume, $V(r,s) = \pi r^2 s$, of a cylinder of radius r and height s is polynomial function of two variables. Its derivative with respect to r is its curved surface area and its derivative with respect to s is its base or lid area. Finally the volume, $V(r,R) = 2\pi^2 r^2 R$, of a torus is a polynomial function of two variables and its derivative with respect to r is its surface area. The common reason for all these facts is given below by considering the example of a circle (*Figure* 1). By definition of the derivative, we have

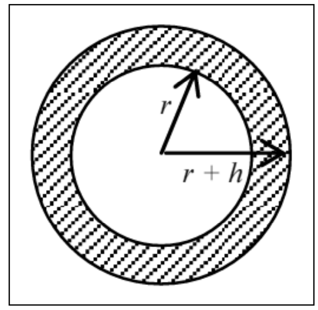


Figure 1.

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}r} &= \lim_{h\to 0}\frac{A(r+h)-A(r)}{h}\\ &= \lim_{h\to 0}\frac{(\text{area of an annulus of width } h)}{h}.\end{aligned}$$

As h approaches zero, the outer boundary of the annulus approaches its inner boundary. And for small values of h, the area of the annulus is equal to the product of the length of its inner boundary (which will be nearly the same as the length of its outer boundary) and its width. To see this, divide the annulus into a large number (say n) of rectangular parts, of width h, by dissecting it with n radii of the outer boundary. The area of each rectangular part is the product of its length (which lies along the inner circle) and height h. So the sum of the areas of all these rectangles is the sum of all the lengths multiplied by their common height h. The sum of all these lengths is the perimeter of the inner circle of the annulus, so the last limit becomes

$$\lim_{h\to 0}\frac{(\text{Perimeter of the inner circle} \times h)}{h} = \text{Perimeter of the inner circle.}$$

Exactly in the same manner, we can show that the derivative of the volume of a sphere of radius r with respect to r is its surface area and derivative of the volume of a cylinder with respect to its radius is its curved surface area etc.

Remark. From this discussion one may conclude that the derivative of the volume of any geometrical shape with respect to a certain variable is its surface area and derivative of the area is its perimeter. But it is not always true as the surface area of an ellipsoid is not the derivative of its volume and the perimeter of an ellipse is not the derivative of its area.

3. Torus

We now compute volume and surface area of the torus and observe that the derivative of its volume is its surface area. We first define the torus precisely.

When a circle in the xz-plane, say $(x - R)^2 + z^2 = r^2$, revolves about the z-axis and completes one revolution, then the surface traced in $\mathbb{R}^3$ is called a torus (a portion of the torus is shown in *Figure* 2). The circles drawn on this torus, which are parallel to the z-axis are called meridians and circles parallel to the xy-plane are called longitudes of the torus.

In order to get the volume and surface area of the torus we cut the torus into slices along the meridian circles. If the number of slices is sufficiently large then each slice

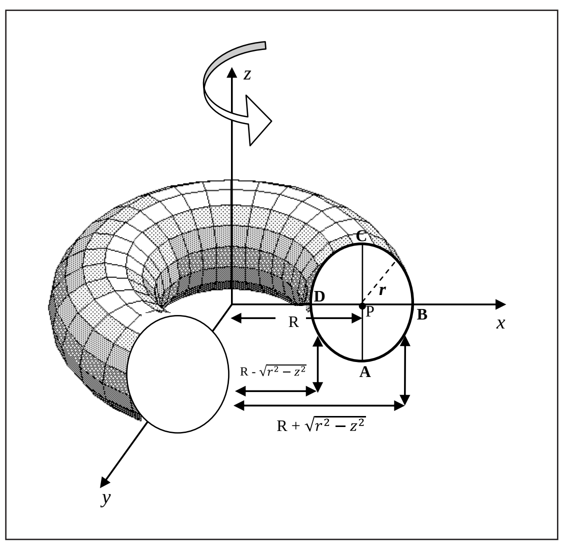


Figure 2.

will look like a right circular cylinder. We know how to compute the volume and surface area of the cylinder. By adding the volumes and curved surface areas of these slices we will get volume and surface area respectively of the torus.

Volume of the Torus. Suppose we divide the torus into k cylinders each of length l; then the volume of each cylinder will be $\pi r^2 l$. The volume of the torus will be the sum of the volumes of these k cylinders, i.e., $\pi r^2 lk$. The product lk is the sum of the lengths of these k cylinders, so it must be equal to $2\pi R$, i.e., the length of the middle longitudinal circle of the torus. Substituting this value of lk we get the volume of the torus to be equal to $2\pi^2 r^2 R$.

Surface Area of the Torus. Again consider the division of the torus into k cylinders each of length l. The curved surface area of each such cylinder will be $2\pi rl$; consequently the surface area of the torus will be $2\pi rlk$, i.e., sum of the curved surface areas of these k cylinders. The product lk is the sum of the lengths of these k cylinders, so it must be equal to $2\pi R$, i.e., the length of the middle longitudinal circle of the torus. Substituting this value of lk we get the surface area of the torus as $4\pi^2 rR$.

Remark. Although we have obtained correct values of the volume and surface area of the torus, we verify it by using calculus because in the proof given here, some approximations have been made that may lead to wrong conclusions.

Torus is a surface of revolution and for such surfaces volume and surface area can be computed by using a well-known Theorem of Pappus, which states that the surface area (volume) of a surface of revolution, generated by rotating a plane curve C (plane surface F) about an external line, is equal to the product of the length of the curve C (area of F) and the distance traveled by its centroid during one revolution. For example:-
Surface Area of the torus = $(2\pi r).(2\pi R)$.

4. Computations Using Calculus

Since a torus is a surface of revolution and for such surfaces there are well-defined techniques in calculus, to compute volume and surface area, we use these techniques as follows.

Volume of the Torus. Notice that volume of the torus is the difference of the volumes of the surfaces of revolution obtained by revolving the arcs ABC and ADC (see *Figure* 2), of the circle $(x - R)^2 + z^2 = r^2$, about the z-axis. Since any horizontal line in the xz-plane, for $-r < z < r$, cuts the arcs ADC and ABC at points $x_1 = R - \sqrt{r^2 - z^2}$ and $x_2 = R + \sqrt{r^2 - z^2}$ respectively, so by using formula for the volume of the surface of revolution, we have

$$\begin{aligned} V &= \int_{-r}^{r} \pi \cdot \left(R + \sqrt{r^2 - z^2}\right)^2 \mathrm{d}z \\ &\qquad\qquad - \int_{-r}^{r} \pi \cdot \left(R - \sqrt{r^2 - z^2}\right)^2 \ \mathrm{d}z \\ &= \pi \cdot \int_{-r}^{r} \left\{\left(R + \sqrt{r^2 - z^2}\right)^2 - \right\} \\ &\qquad\qquad \left\{\left(R - \sqrt{r^2 - z^2}\right)^2\right\} \ \mathrm{d}z \\ &= 4\pi R \cdot \int_{-r}^{r} \sqrt{r^2 - z^2} \ \mathrm{d}z \\ &= 2\pi^2 r^2 R\,. \end{aligned}$$

Surface Area of the Torus. This can be obtained by adding the surface areas of the surfaces of revolution obtained by revolving the arcs ABC and ADC about the z-axis (see *Figure* 2). We know that if a curve $x = g(z)$ is rotated about the z-axis, then the surface area of the surface of revolution thus obtained, can be computed from the following formula:

$$\int_{a}^{b} 2\pi \cdot \left[g(z) \cdot \sqrt{1 + \{g'(z)\}^2}\right] \mathrm{d}z\,.$$

By applying this formula on the curves ABC and ADC we get

Surface area

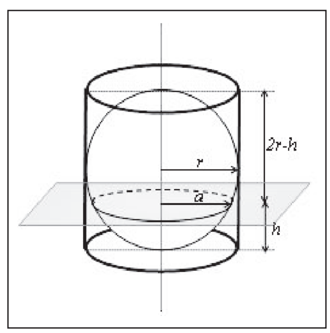
$$
\begin{aligned}
&= \int_{-r}^{r} 2\pi \cdot \left[\left\{ R + \sqrt{r^2 - z^2} \right\} \cdot \sqrt{1 + \frac{z^2}{r^2 - z^2}} + \right. \\
&\qquad \left. \left\{ R - \sqrt{r^2 - z^2} \right\} \cdot \sqrt{1 + \frac{z^2}{r^2 - z^2}} \right] \, \mathrm{d}z \\
&= \int_{-r}^{r} 2\pi \cdot \left[\frac{r}{\sqrt{r^2 - z^2}} \cdot 2R \right] \, \mathrm{d}z \\
&= 4\pi r R \cdot \int_{-r}^{r} \frac{1}{\sqrt{r^2 - z^2}} \, \mathrm{d}z \\
&= 4\pi^2 r R.
\end{aligned}
$$

5. A Deduction from the Archimedes' Sphere–Cylinder Theorem

Archimedes showed that the surface area of a sphere of radius r is equal to the curved surface area of a circumscribing cylinder of radius r and height $2r$. In fact he proved the following more general result: 'If a cylinder – circumscribing a sphere – is cut by a plane parallel to its base then the slice of the cylinder and the sphere underneath this plane will have same surface areas'. The result is known as Archimedes' Sphere–Cylinder Theorem. Archimedes considered this result as his most beautiful and important achievement. He desired that his favourite picture of sphere and cylinder be carved on his grave after his death and his desire was fulfilled.

By using Archimedes' result we shall compute the volume of the slice of the sphere cut out by a plane at a height h from the ground as shown in *Figure* 3.

Figure 3.



Assume that the boundary circle of the slice of the sphere of radius r, has radius a and also assume that m triangles (m is sufficiently large) have been drawn on this slice. If we join the vertices of these triangles with the center of the sphere, we get m cones, each having a triangular base and height r. Notice that the sum of the

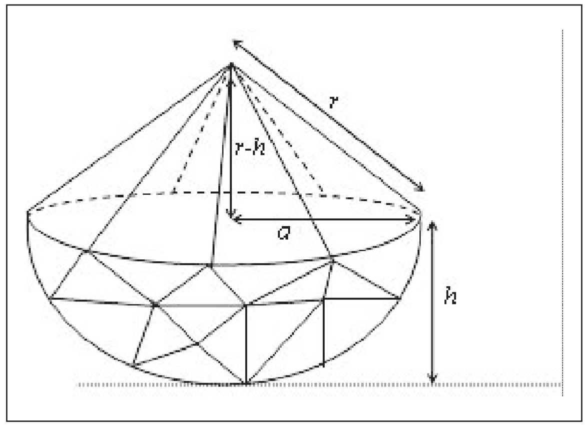


Figure 4.

volumes of these m cones minus the volume of a cone with base radius a and height $r-h$ gives the volume of the required slice (see *Figure* 4).

Required volume =

$$\sum_{i=1}^{m}(\frac{1}{3}\text{area of } i\text{th triangle} \times r) - \frac{1}{3}\pi a^2(r-h)$$

$$= \frac{1}{3}\{(\text{surface area of the sphere's slice}) \times r - \pi a^2(r-h)\}$$

$$= \frac{1}{3}\{(\text{surface area of the cylinder's slice}) \times r - \pi a^2(r-h)\}$$

(by using Sphere–Cylinder Theorem)

$$= \frac{1}{3}\{(2\pi rh) \times r - \pi a^2(r-h)\}$$

$$= \frac{1}{3}\{2\pi r^2 h - \pi a^2(r-h)\}. \quad (1)$$

From *Figure* 4, it follows that $a^2 + (r-h)^2 = r^2$

$$\Rightarrow a^2 = r^2 - (r-h)^2 = h(2r-h). \quad (2)$$

From (1) and (2) we get

Volume of the sphere's slice

$$\begin{aligned} &= \frac{1}{3}\{2\pi r^2 h - \pi h(2r-h)(r-h)\} \\ &= \frac{1}{3}\pi\{2r^2 h + 2rh^2 - h^3 - 2r^2 h + rh^2)\} \\ &= \frac{1}{3}\pi h^2(3r-h) \qquad (3) \end{aligned}$$

Corollary. Substituting $h = 2r$ in equation (3) proves that the volume of the sphere is $\frac{4}{3}\pi r^3$.

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