Combinatorial Proofs and Algebraic Proofs – I

Shailesh A Shirali

Shailesh A Shirali is Director of Sahyadri School (KFI), Pune, and also Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been in the field of mathematics education for three decades, and has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books addressed to high school students, and serves as an editor for *Resonance* and for *At Right Angles*. He is engaged in many outreach projects in teacher education.

It is sometimes the case in mathematics that the same result can be proved in two or more essentially different ways (a luxury of mathematical riches, you might say). A recurrent theme in this context is the tension between *combinatorial proofs* and *algebraic proofs*. In this two-part article we describe some results which allow for proofs of both kinds and give us an opportunity to contrast them. We shall see some beautiful examples of both kinds of proofs. In Part I we dwell on results that deal with the binomial coefficients and with Fibonacci numbers.

We will not be overly zealous in our usage of the term 'combinatorial': we will allow a proof to be so categorized if it is *largely* combinatorial, with only a 'small' component of algebra. However, that small part may well be an essential component of the proof.

The simplicity of counting arguments can be deceptive. In recent times, Paul Erdős has shown just how subtle and powerful such arguments can be, and how far they can reach.

1. Introducing the Theme

We introduce the theme by giving combinatorial proofs of two simple identities:

$$n^2 = (n-1)^2 + 2(n-1) + 1, \tag{1}$$

and

$$\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + mn. \tag{2}$$

Keywords
Combinatorial proof, algebraic proof, binomial identity, recurrence relation, composition, Fibonacci number, Fibonacci identity, Pascal triangle.

For (1) we count the number of pairs (i, j) where $1 \leq i, j \leq n$. The number is clearly n^2. Now subdivide the pairs according to how many 1s they have. There are clearly $(n-1)^2$ pairs with no 1; $2(n-1)$ pairs with one 1; and just 1 pair with two 1s. Hence the stated identity. The identity $n^3 = (n-1)^3 + 3(n-1)^2 + 3(n-1) + 1$ follows in exactly the same way.

For identity (2), we count the number of ways of selecting two elements from a set S containing $m+n$ elements. The number is clearly $\binom{m+n}{2}$. Now let S be arbitrarily partitioned into two subsets A and B with m elements and n elements, respectively. Either both the selected elements are from A, or both are from B, or there is one each from A and B. Accounting for these three possibilities we get (2).

The two proofs illustrate the basic strategy followed in many combinatorial proofs: that of counting the same quantity in two or more different ways, and equating the resulting expressions.

The basic strategy followed in such proofs is that of counting the same quantity in two different ways.

Remark. By counting the number of pairs (i, j) where $1 \leq i, j \leq n$ in another way, we get a different identity. Classify the pairs according to the value of $\max(i, j)$. How many pairs have a 'max value' of 1? Clearly 1; the only such pair is $(1, 1)$. How many pairs have a max value of 2? Clearly 3, as there are three such pairs: $(1, 2)$, $(2, 1)$ and $(2, 2)$. Similarly there are 5 pairs with max value 3; there are 7 pairs with max value 4; and $(2n-1)$ pairs with max value n. Since there are n^2 pairs in all we have arrived at the identity

$$1 + 3 + 5 + \cdots + (2n-1) = n^2. \qquad (3)$$

2. Binomial Identities

Next we present some identities involving the binomial coefficients. These may be proved using either the binomial theorem or by combinatorial arguments; but we present only the latter. The prototypical example of

We get this identity by counting in two ways the subsets of a set with *n* elements.

such identities is the identity

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n,$$

which we get from counting in two different ways the subsets of a set with n elements.

2.1 *Committees*

We prove the following: for any positive integer n:

$$1\cdot\binom{n}{1}+2\cdot\binom{n}{2}+3\cdot\binom{n}{3}+\cdots=n\cdot 2^{n-1}. \tag{4}$$

We do so by counting committees. *Given n persons, in how many ways can we select a committee from the group*? (The size of the committee is not specified; it could be any number from 1 to n.) We take for granted the understanding that a committee must have a Chairperson.

We may opt to fix the size k of the committee first, select the k members in $\binom{n}{k}$ ways, and then appoint a Chairperson from among the selected members; k choices. This gives a total of $\sum_{k=1}^{n} k\cdot\binom{n}{k}$ ways of selecting the committee.

Or we may *first* select the Chairperson (n choices), and *then* select the remaining members in 2^{n-1} ways (any subset of the remaining $n-1$ members may be chosen). This gives a total of $n\cdot 2^{n-1}$ ways of selecting a committee. Equality (4) follows.

Arguing the same way we may establish the following:

$$\sum_{k=1}^{n} k(k-1)\cdot\binom{n}{k}=n(n-1)\cdot 2^{n-2}. \tag{5}$$

From (4) and (5) we obtain a third one:

$$\sum_{k=1}^{n} k^2\cdot\binom{n}{k}=n(n+1)\cdot 2^{n-2}. \tag{6}$$

A combinatorial interpretation for this identity is not so easy to find (although there exists one)! Another such is the following (we ask the reader to find a similar suitable interpretation):

$$\sum_{k=1}^{n} k^3 \cdot \binom{n}{k} = n^2(n+3) \cdot 2^{n-3}. \qquad (7)$$

2.2 *Odd Subsets, Even Subsets*

Let S be a set with n elements ($n \geq 1$). We shall show: *The number of subsets of S with an odd number of elements is equal to the number of subsets with an even number of elements.* (We refer to such subsets as 'odd subsets' and 'even subsets' respectively.)

The number of odd subsets equals the number of even subsets.

Let x be any *fixed* element of S. We construct a two-way classification of all the subsets of S as follows:

	Odd cardinality	Even cardinality
Containing x	O_1	E_1
Not containing x	O_2	E_2

We establish a 1–1 correspondence between O_2 and E_1 as follows: if a subset $T \subset S$ is of type O_2, then $T \cup \{x\}$ is of type E_1. Conversely, if $T \subset S$ is of type E_1, then $T \setminus \{x\}$ is of type O_2. This bijection shows that $|O_2| = |E_1|$. A similar bijection shows that $|O_1| = |E_2$. Hence, there are as many odd subsets as even subsets.

2.3 *Subsets Mod 4*

Let S be an arbitrary set with n elements. For $i \in \{0, 1, 2, 3\}$, let $f(n, i)$ denote the number of subsets $T \subset S$ such that $|T| \equiv i \pmod 4$. In analogy with the recurrence relation $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ we have the following:

$$f(n, i) = f(n-1, i) + f(n-1, i-1), \qquad (8)$$

where arithmetic for the second argument of f is done modulo 4. (To see why, consider whether any fixed element, say n, belongs to the subset T or not. Since

$f(n-1, i)$ is the number of subsets T for which $n \notin T$, while $f(n-1, i-1)$ is the number of subsets T for which $n \in T$, relation (8) follows.)

The average of $f(n, 0)$, $f(n, 1)$, $f(n, 2)$ and $f(n, 3)$ being 2^{n-2}, we are motivated to study the quantity $g(n, i) = f(n, i) - 2^{n-2}$. Shown below are the values taken by $g(n, i)$ for $2 \leq n \leq 12$.

n	$g(n,0)$	$g(n,1)$	$g(n,2)$	$g(n,3)$
2	0	1	0	-1
3	-1	1	1	-1
4	-2	0	2	0
5	-2	-2	2	2
6	0	-4	0	4
7	4	-4	-4	4
8	8	0	-8	0
9	8	8	-8	-8
10	0	16	0	-16
11	-16	16	16	-16
12	-32	0	32	0

On examining such data we see numerous striking patterns. Here are some of the many relations, which all have a similar 'look':

1. *If* $n \equiv 0 \pmod 4$ *then* $f(n, 1) = 2^{n-2}$.
2. *If* $n \equiv 2 \pmod 4$ *then* $f(n, 0) = 2^{n-2}$.
3. *If* $n \equiv 0 \pmod 8$ *then* $f(n, 0) = 2^{n-2} + 2^{(n-2)/2}$.
4. *If* $n \equiv 4 \pmod 8$ *then* $f(n, 0) = 2^{n-2} - 2^{(n-2)/2}$.

Can we account for these patterns combinatorially? The first two are easy to handle, but the next two are far from clear; we give only an algebraic proof of the third one.

Proof of (1). We know that $f(n,0)+f(n,2) = 2^{n-1} = f(n,1) + f(n,3)$; so it suffices to show that $f(n,1) = f(n,3)$. The following bijection does the needful: *Map each subset to its complement.* For, since $n \equiv 0 \pmod 4$, if a subset T has cardinality $i \pmod 4$, its complement $T' = S \setminus T$ has cardinality $-i \pmod 4$; and conversely. The result follows.

By mapping each subset of a given set to its complement, interesting identities can be obtained.

Proof of (2). It suffices to show that $f(n,0) = f(n,2)$. The very same bijection works: since $n \equiv 2 \pmod 4$, if a subset T has cardinality $i \pmod 4$, its complement $T' = S \setminus T$ has cardinality $2-i \pmod 4$; and conversely. The result follows.

Proof of (3). As already emphasized, we give only an algebraic proof. Consider the binomial expansions of $(1 \pm i)^n$, where $i = \sqrt{-1}$; we have:

$$\binom{n}{0} \pm i \cdot \binom{n}{1} + i^2 \cdot \binom{n}{2} \pm i^3 \cdot \binom{n}{3} + \cdots = (1 \pm i)^n.$$

By addition we get:

$$\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \cdots = \frac{(1+i)^n + (1-i)^n}{2}.$$

Since n is a multiple of 8, and $(1 \pm i)^8 = 2^4$, the quantity on the right simplifies to $2^{n/2}$. Hence $\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \cdots = 2^{n/2}$. On the other hand, $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1}$. Hence:

$$\binom{n}{0} + \binom{n}{4} + \binom{n}{8} + \cdots = \frac{2^{n-1} + 2^{n/2}}{2} = 2^{n-2} + 2^{(n-2)/2},$$

as claimed. Equality (4) may be proved along the same lines.

2.4 *Subsets Mod 3*

When we study the corresponding situation modulo 3, we find a different pattern, simpler and more transparent; and this time a neat combinatorial proof is at hand, based on recurrence relations. (So we use a 'bit' of algebra.)

Let S be an arbitrary set with n elements. For $i \in \{0, 1, 2\}$, let $h(n, i)$ denote the number of subsets $T \subset S$ such that $|T| \equiv i \pmod 3$. Shown below are the values taken by $h(n, i)$ for $1 \leq n \leq 10$.

n	$h(n,0)$	$h(n,1)$	$h(n,2)$
2	1	2	1
3	2	3	3
4	5	5	6
5	11	10	11
6	22	21	21
7	43	43	42
8	85	86	85
9	170	171	171
10	341	341	342

We notice the following right away: *The set* $\{h(n, 0), h(n, 1), h(n, 2)\}$ *consists of a pair of consecutive numbers*. Otherwise put: The differences between the numbers $h(n, 0)$, $h(n, 1)$, $h(n, 2)$ are 0, 1, 1 in some order. (Thus, the three quantities are 'almost' the same. Recall that we had a very different situation modulo 4.) Denote the statement by P_n.

We shall prove P_n using the recurrence relation connecting the $h(n, i)$:

$$h(n, i) = h(n - 1, i) + h(n - 1, i - 1), \qquad (9)$$

where arithmetic for the second argument of h is done modulo 3. From (9) we get:

$$\begin{cases} h(n, 0) - h(n, 1) \ = \ h(n - 1, 2) - h(n - 1, 1), \\ h(n, 1) - h(n, 2) \ = \ h(n - 1, 0) - h(n - 1, 2), \\ h(n, 2) - h(n, 0) \ = \ h(n - 1, 1) - h(n - 1, 0). \end{cases}$$

Thus, if P_{n-1} is true, so is P_n. Since P_1 is true, it follows that P_n is true for every n.

This observation allows us to compute the exact value of $h(n,i)$ for each n and i. For example, suppose that n is a multiple of 3. By mapping each subset of S to its complement, we find that $h(n,1) = h(n,2)$. We cannot have $h(n,0) = h(n,1)$, as that would mean that the sum $h(n,0) + h(n,1) + h(n,2)$ is a multiple of 3, which cannot be as the sum is 2^n. Hence we have $h(n,0) = a$, $h(n,1) = b$, $h(n,2) = b$, where $a = b \pm 1$. This yields the equality $2^n = 3b \pm 1$. Since $2^n \equiv -1 \pmod 3$ when n is odd, and $2^n \equiv 1 \pmod 3$ when n is even, we deduce that $a = b - 1$ if n is an odd multiple of 3, and $a = b + 1$ if n is an even multiple of 3. Therefore: if n is an odd multiple of 3,

$$h(n,0) = \frac{2^n - 2}{3}, \quad h(n,1) = \frac{2^n + 1}{3}, \quad h(n,2) = \frac{2^n + 1}{3},$$

while if n is an even multiple of 3,

$$h(n,0) = \frac{2^n + 2}{3}, \quad h(n,1) = \frac{2^n - 1}{3}, \quad h(n,2) = \frac{2^n - 1}{3}.$$

Similar results may be obtained for $n \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$. The results show an underlying periodic pattern with a cycle length of 6. More specifically, if $k(n,i) = 3\,h(n,i) - 2^n$, then $k(n+6,i) = k(n,i)$.

3. Compositions

By a *composition* (also called an *ordered partition*) of a positive integer n we mean a finite sequence of positive integers whose sum is n. Here are the compositions of 4, shown as tuples:

A composition is a bit like a partition; but order is taken into account.

$$(4),\ (3,1),\ (1,3),\ (2,2),\ (2,1,1),\ (1,2,1),\ (1,1,2),\ (1,1,1,1).$$

We show the following: *n has 2^{n-1} compositions.*

We first give a proof using recursion and a bit of algebra. Let a_n denote the number of compositions of n. The only composition of n with just one summand is (n); we store this away for the moment. If there is more than one summand, then the first number used is some number k, where $1 \leq k \leq n-1$. The summands following this number constitute a composition of $n-k$. Letting k take all possible values, and recalling the singleton composition 'stored' away, we see that

$$a_n = 1 + a_1 + a_2 + \cdots + a_{n-1}. \qquad (10)$$

Relation (10) yields $a_{n-1} = 1 + a_1 + a_2 + \cdots + a_{n-2}$, and from these we deduce that

$$a_n = 2 \cdot a_{n-1}. \qquad (11)$$

Since $a_1 = 1 = 2^0$, we get $a_n = 2^{n-1}$. Note the roles played by combinatorial thinking and algebra in this proof: both parts are essential. (Importantly, even if we had not guessed the formula for a_n, we could get it using the recursive relationship.)

In contrast, here is a combinatorial proof, which we construct *after* we get to know the formula for a_n. Since 2^{n-1} counts the number of subsets of $\{1, 2, 3, \ldots, n-1\}$, it is natural to ask: *Is there a natural bijection between the compositions of* n *and the subsets of* $\{1, 2, 3, \ldots, n-1\}$? There is, and it is easy to find.

Figure 1 shows n dots in a row, with adjacent pairs separated by vertical lines ('fences'); there are $n-1$ such fences, and they have been numbered 1 to $n-1$. We may think of the picture as showing n dots housed in various 'rooms' separated by the fences. Now the desired bijection is easily found. Let any subset of the

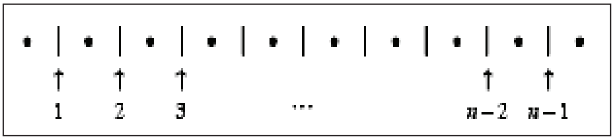

Figure 1. Counting the compositions of *n*.

fences be removed. Then the fences that remain create rooms in a natural way and thus define a composition of n; we simply count the number of dots in the various rooms. For example, if we remove all the fences, we get the composition made up of a solitary n; and if we do not remove any of them, we get the composition which has only 1s.

4. Compositions and Fibonacci Numbers

Let us modify the above setting slightly: *we will not allow* 1 *to be used as a summand in the composition.* We get a very different answer now — but it is just as pleasing! We find that 4 has two such compositions, (4) and $(2, 2)$, while 5 has three such: (5), $(3, 2)$ and $(2, 3)$. We prove the following: *The number of compositions of n in which every summand exceeds* 1 *is F_{n-1}, where F_k denotes the k^{th} Fibonacci number.* (Recall that $F_0 = 0$, $F_1 = 1$, $F_k = F_{k-1} + F_{k-2}$ for $k \geq 2$.)

It would not be reasonable now to expect to find a *purely* combinatorial proof — because the Fibonacci numbers are arithmetically defined; the definition *per se* has no combinatorial connotation. So now we allow a pinch of algebra.

The Fibonacci numbers are defined arithmetically; the definition does not as such have a combinatorial meaning.

Let b_n be the number of compositions of n in which every summand exceeds 1; then $b_1 = 0$, $b_2 = 1$, $b_3 = 1$. We now argue in the same way as earlier. Assume that $n \geq 2$. Put aside the composition in which there is just one summand. Let the first number used be k, where $2 \leq k \leq n - 2$. The summands following this number constitute a composition of $n - k$. Letting k take all possible values, and recalling the singleton composition stored away, we see that

$$b_n = 1 + b_1 + b_2 + \cdots + b_{n-2}. \qquad (12)$$

Relation (12) yields $b_{n-1} = 1 + b_1 + b_2 + \cdots + b_{n-3}$, and from these we deduce that

$$b_n = b_{n-1} + b_{n-2}. \qquad (13)$$

Thus these numbers follow the same recursion as the Fibonacci numbers. And since $b_1 = 0$, $b_2 = 1$, $b_3 = 1$, whereas $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, the two sequences have the same numbers, but with an offset of 1 between them. Hence $b_n = F_{n-1}$.

The n-th Fibonacci number is equal to the number of compositions of $n + 1$ in which every summand exceeds 1.

As noted earlier, the Fibonacci numbers are not combinatorially defined. But the above result allows us (if we so wish) to fashion such a definition: *F_n is the number of compositions of $n + 1$ in which every summand exceeds* 1.

4.1 *Fibonacci Again*

Now we consider compositions of n whose summands are *all odd*; let their number be c_n. Here are some data for a few small values of n.

n	Compositions using odd summands only	c_n
1	(1)	1
2	$(1,1)$	1
3	$(3),\ (1,1,1)$	2
4	$(3,1),\ (1,3),\ (1,1,1)$	3
5	$(5),\ (3,1,1),\ (1,3,1),\ (1,1,3),\ (1,1,1,1,1)$	5

The reader is invited to verify that $c_6 = 8$ and $c_7 = 13$. Now we find: $c_n = F_n$.

The proof is yet again by recursion. Empirically we get: $c_1 = 1$, $c_2 = 1$, so $c_n = F_n$ for $n \leq 2$. Since the last entry of an 'odd' composition must be one of the numbers 1, 3, 5, 7, ... we get:

$$c_n = c_{n-1} + c_{n-3} + c_{n-5} + c_{n-7} + \cdots . \qquad (14)$$

Replacing n by $n - 2$ throughout, we get $c_{n-2} = c_{n-3} + c_{n-5} + c_{n-7} + \cdots$. This yields $c_n = c_{n-1} + c_{n-2}$, implying that these numbers follow the very same recursion as the Fibonacci numbers. As the sequences agree in the first two terms, they are identical.

Now we have another combinatorial definition of the Fibonacci numbers, more appealing and compact than the earlier one: *F_n is the number of compositions of n that use only odd summands.*

The n-th Fibonacci number also equals the number of compositions of n that use only odd summands.

4.2 *Fibonacci Yet Again*

Remarkably, the Fibonacci numbers feature in yet another problem about compositions. This time we consider compositions of n in which *the summands are all* 1*s and* 2*s*. Let d_n denote the number of such compositions of n. For example, the compositions of 2 under this constraint are (2) and (1, 1), while the compositions of 3 are (2, 1), (1, 2) and (1, 1, 1). Thus $d_1 = 1$, $d_2 = 2$, $d_3 = 3$, $\ldots$. The connection now is: $d_n = F_{n+1}$. It is easy to establish.

Consider the first summand. If it is 1, then the remaining summands constitute a composition of $n - 1$ using only 1s and 2s, and if it is 2, then the summands constitute a composition of $n - 2$ under the same constraint. Hence $d_n = d_{n-1} + d_{n-2}$, and the rest follows as earlier.

4.3 *Tilings*–1

The fact that the number of compositions of n using only 1s and 2s as summands is F_{n+1} may be expressed in a visually more appealing way in terms of tilings. We are given a $1 \times n$ rectangular strip which we pack using unit squares and dominoes (rectangles of size 1×2). The number of ways of doing this is clearly $d_n = F_{n+1}$. *Figure* 2 shows three such tilings of a 1×5 rectangle, corresponding to the compositions $1 + 1 + 1 + 1 + 1$, $1 + 2 + 2$ and $2 + 1 + 1 + 1$.

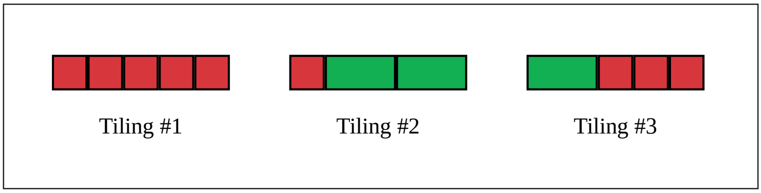

Figure 2. Three tilings of a 1×5 rectangle using unit squares and dominoes.

The number of tilings using dominoes is yet again a Fibonacci number.

4.4 *Tilings–2*

If we consider rectangles of size $2 \times n$, the number of tilings using dominoes alone is yet again a Fibonacci number. For, if e_n denotes the number of such tilings, then $e_1 = 1$, $e_2 = 2$, and $e_n = e_{n-1} + e_{n-2}$ (to see why, focus attention on the last two columns of the rectangle); hence $e_n = F_{n+1}$. (Or, more simply: by bisecting the rectangle with a horizontal line we get $e_n = d_n$; for, the two halves must be identical, giving us two copies of a tiling of a $1 \times n$ rectangle using unit squares and dominoes.) *Figure* 3 shows the five possible domino tilings of a 2×4 rectangle.

4.5 *A Query*

On noticing that $b_{n+2} = c_{n+1} = d_n$, we are provoked to ask: *Is there a natural bijection between the compositions of* $n + 2$ *using summands exceeding* 1*, the compositions of* $n+1$ *using odd summands, and the compositions of* n *using* 1*s and* 2*s*? This problem appears to be difficult; I do not have an answer.

5. Some Fibonacci Identities

We know that the number of compositions of n using 1s and 2s is F_{n+1}. We now show that this observation leads to some identities involving the Fibonacci numbers.

Consider the compositions of $n+1$ with only 1s and 2s, and having *at least one* 2. They must number $F_{n+1} - 1$, as there is just one composition made up of only 1s.

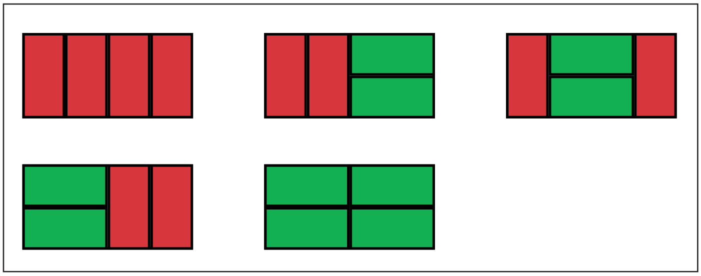

Figure 3. Tilings of a 1×4 rectangle using dominoes.

Such a composition has a 'last' 2. Let there be k occurrences of 1 after this 2, where naturally $0 \leq k \leq n-1$. The composition then has the following appearance:

$$(\underbrace{\ldots,,\ \ldots,\ \ldots,\ \ldots,\ \ldots}_{\text{Mixture of 1s and 2s}},\ 2,\ \underbrace{1,\ 1,\ 1,\ \ldots,\ 1}_{k\ \text{1s}})$$

In this depiction, the 'mixture' constitutes a composition of $n-k-2$. As every composition of $n-k-2$ will yield one such picture, we get the following identity:

$$F_{n+1} - 1 = F_0 + F_1 + F_2 + \cdots + F_{n-1}. \qquad (15)$$

A slight change of perspective yields another identity. Assume that n is even; then the composition has at least one 1. We look at the position of the last 1. Let there be k occurrences of 2 after the last 1, where $k \geq 0$. The numbers before the 1 constitute a composition of $n-2k-1$, and there are F_{n-2k} such compositions. Hence we get $F_{n+1} = \sum_{k\geq 0} F_{n-2k}$, for even values of n. Or, changing the indexing slightly:

$$F_{2n+1} = F_0 + F_2 + F_4 + \cdots + F_{2n}. \qquad (16)$$

Identities (15) and (16) are generally proved using induction. Though these proofs are easy, it is still of interest to find these combinatorial interpretations.

5.1 *A Slant Pascal Connection*

Yet another identity may be found by considering the *number* of 2s used. Let this number be k (obviously, $k \leq \frac{1}{2}n$); then the number of 1s is $n-2k$. The length of the tuple is thus $k+n-2k = n-k$. The 2s may occupy any k places in this tuple. Hence:

$$F_{n+1} = \sum_{k=0}^{n/2} \binom{n-k}{k}.$$

That is:

$$F_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots. \qquad (17)$$

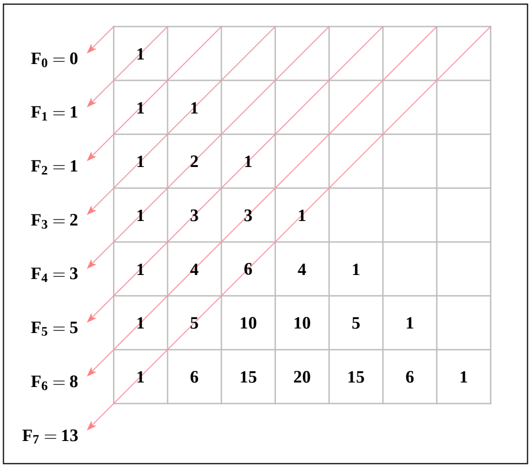

Figure 4. *Diagonal sums on the Pascal triangle.*

This identity establishes a 'slant connection' between the Fibonacci numbers and the Pascal triangle. See *Figure* 4.

It is more difficult to find combinatorial proofs of second-order identities.

5.2 *A Second Order Fibonacci Identity*

Generally speaking, it is more difficult to find combinatorial proofs of second order identities. For example, it is a challenge to find such a proof for the relation $F_n^2 - F_{n-1}F_{n+1} = \pm 1$, which on the other hand is easy to prove using induction, or the Binet formula. Here, we consider the following relation:

$$F_n^2 + F_{n+1}^2 = F_{2n+1}. \tag{18}$$

For example, for $n = 3$ we have: $F_3^2 + F_4^2 = 2^2 + 3^2 = 13 = F_7$.

For the proof we define the notion of a 'gap' in a composition. Consider the tiling of a $1 \times n$ rectangle corresponding to a composition of n using 1s and 2s; the gaps are then the vertical grid lines which are *missing* from the tiling. The pictures in *Figure* 5 illustrate this notion better than our description: in the second tiling the gaps are 2 and 4, and in the third one there is just one gap, 1. Note that a number i is a gap in a given composition precisely when there is a domino 'sitting' upon it.

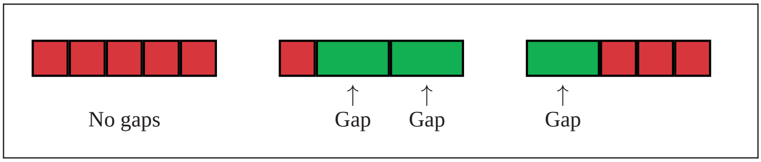

Figure 5. Notion of a gap in a tiling, shown for the case $n = 5$.

Let us classify the compositions of $2n$ according to whether n is a gap or not. If n is a gap, then neither $n-1$ nor $n+1$ is a gap. Hence the composition has the following appearance: a composition of $n-1$ using 1s and 2s, followed by a 2, followed by another composition of $n-1$ using 1s and 2s (possibly the same one). Therefore the number of such compositions of $2n$ in which n is a gap is F_n^2. On the other hand, if n is not a gap, then the composition has the following appearance: a composition of n using 1s and 2s, followed by another composition of n using 1s and 2s (possibly the same one). Hence the number of such compositions of $2n$ in which n is not a gap is F_{n+1}^2. It follows that $F_n^2 + F_{n+1}^2 = F_{2n+1}$.

Suggested Reading

[1] **A T Benjamin and J Quinn, *Proofs That Really Count: The Art of Combinatorial Proof*, MAA, 2003.**
This book is highly recommended – it has a rich collection of combinatorial proofs. The notion of a 'gap' of a tiling is taken from this book.

[2] **http://en.wikipedia.org/wiki/Combinatorial_proof**

[3] **http://mathforum.org/library/drmath/view/56142.html**

Address for Correspondence
Shailesh A Shirali
Sahyadri School
Tiwai Hill, Rajgurunagar
Pune 410 513
Email:
shailesh.shirali@gmail.com