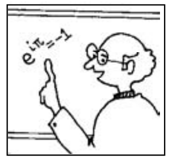
Classroom



In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. “Classroom” is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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The Condition for Minimum Deviation through Prism
Two Proofs without using Calculus

We know that the condition for minimum deviation of a ray passing through a prism is that the angle of incidence is equal to the angle of emergence. It is generally proved by using differential calculus. Here two proofs have been provided without using calculus, the first by application of trigonometry only and the second by pure reasoning.

Keywords
Prism, minimum deviation, angle of incidence, angle of emergence.

This is an age-old problem which figures in undergraduate and senior secondary textbooks on geometrical optics.

First Proof

Let us first consider the direction of a ray at a single refracting surface due to refraction (*Figure* 1).

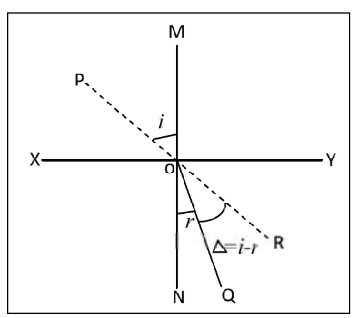

Figure 1. Deviation at a single surface.

XY is a section of the surface of separation between two media. O is the point of incidence and MON is the normal. PO is the incident ray and OQ the refracted ray.

$\angle$ MOP $= i =$ angle of incidence ,
$\angle$ QON$= r =$ angle of refraction ,
$\angle$ ROQ $=$ the angle of deviation $= \Delta = i - r$.

We know that

$$\frac{\sin i}{\sin r} = \mu \,,$$
$$\text{or } \sin i = \mu \sin r \qquad (1)$$

$$\therefore \sin i - \sin r = (\mu - 1) \sin r$$

$$\text{or } 2\cos\left(\frac{i+r}{2}\right)\sin\left(\frac{i-r}{2}\right) = (\mu - 1)\,\sin r$$
$$\text{or } \sin\left(\frac{i-r}{2}\right) = \frac{(\mu-1)\,\sin r}{2\cos\left(\frac{i+r}{2}\right)}$$
$$\text{or } \sin\frac{\Delta}{2} = \frac{(\mu-1)\,\sin r}{2\cos\left(\frac{i+r}{2}\right)}\,. \qquad (2)$$

From (1) we get that when 'i' increases, r increases, and therefore $\sin r$ increases. Again $(i + r)$ increases, i.e., $\frac{i+r}{2}$ increases. So $\cos(\frac{i+r}{2})$ decreases. Thus with increase of 'i', the numerator of the RHS of (2) increases and the denominator decreases. So the quantity as a whole increases. In other words, $\sin(\frac{\Delta}{2})$ increases with increase of 'i', which means that Δ increases with increase of 'i'.

Now, let us come to the situation in a prism (*Figure* 2).

ABC is the principal section of a prism. DE, EF and FG are respectively the incident, refracted and the emergent rays. The total deviation $\delta = \angle$ KJG.

$$\delta = (i_1 - r_1) + (i_2 - r_2)\,,$$
$$\delta = (i_1 + i_2) - (r_1 + r_2)\,. \qquad (3)$$

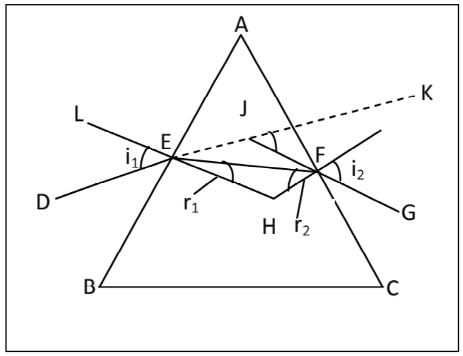

Figure 2. Refracton through a prism.

We see from ΔEFH that

$$\angle\ \mathrm{HEF} + \angle\ \mathrm{EFH} + \angle\ \mathrm{EHF} = 180^{\mathrm{o}}\,. \tag{4}$$

Again, as $\angle$ AEH $=$ $\angle$ AFH $=$ 90°, the quadrilateral AFHE is cyclic, So,

$$\angle\ \mathrm{EAF} + \angle\ \mathrm{EHF} = 180^{\mathrm{o}}\,. \tag{5}$$

From (4) and (5),

$$\angle\ \mathrm{HEF} + \angle\ \mathrm{EFH} = \angle\ \mathrm{EAH} \tag{6}$$

$$\text{or} \quad r_1 + r_2 = A\,. \tag{7}$$

$\therefore$ From (3) and (6)

$$\delta = i_1 + i_2 - A\ .$$

Now, from Snell's law, we know that

$$\mu = \frac{\sin i_1}{\sin r_1} = \frac{\sin i_2}{\sin r_2} = \frac{\sin i_1 + \sin i_2}{\sin r_1 + \sin r_2}$$

$$\text{or } \mu = \frac{2\sin\left(\frac{i_1+i_2}{2}\right)\cos\left(\frac{i_1-i_2}{2}\right)}{2\sin\left(\frac{r_1+r_2}{2}\right)\cos\left(\frac{r_1-r_2}{2}\right)}$$

$$\therefore \mu = \frac{\sin\left(\frac{i_1+i_2}{2}\right)}{\sin\left(\frac{r_1+r_2}{2}\right)} \times \frac{\cos\left(\frac{i_1-i_2}{2}\right)}{\cos\left(\frac{r_1-r_2}{2}\right)}$$

$$\therefore \mu = \left(\frac{\sin\frac{\delta+A}{2}}{\sin\frac{A}{2}}\right) \times f,$$

where

$$f = \frac{\cos\left(\frac{i_1 - i_2}{2}\right)}{\cos\left(\frac{r_1 - r_2}{2}\right)} .$$

Now, $\mu \sin \frac{A}{2}$ = a constant. Therefore $\sin\left(\frac{\delta + A}{2}\right)$ will be minimum when f is maximum, i.e., δ will be mimimum when f is maximum. So let us find the condition for f to be maximum. For that, we shall explore three cases which are exhaustive:

$$\text{(a) } i_1 > i_2, \ \text{(b) } i_1 < i_2 \text{ and (c) } i_1 = i_2 .$$

Case (a) : $i_1 > i_2$.
Since $i_1 > i_2$, the respective value of Δ for i_1 will be greater than that for i_2, which means that

Cos q decreases with q for 0° < q < 180°.

$$i_1 - r_1 > i_2 - r_2$$

$$\text{or,} \quad i_1 - i_2 > r_1 - r_2$$

$$\therefore \frac{i_1 - i_2}{2} > \frac{r_1 - r_2}{2}$$

$$\therefore \cos\left(\frac{i_1 - i_2}{2}\right) < \cos\left(\frac{r_1 - r_2}{2}\right)$$

$$\therefore f < 1,$$

Case (b) : $i_1 < i_2$.
In this case, $i_1 - r_1 < i_2 - r_2$

Cos q increases with q for –180° < q < 0°.

$$\text{or } i_1 - i_2 < r_1 - r_2$$

$$\therefore \frac{i_1 - i_2}{2} < \frac{r_1 - r_2}{2} .$$

Now, here is a catch: the quantities on the two sides of the 'less than' (<) sign are negative.

$$\therefore \cos\left(\frac{i_1 - i_2}{2}\right) < \cos\left(\frac{r_1 - r_2}{2}\right)$$

$$\therefore f < 1, \text{again} .$$

Case (c) : $i_1 = i_2$.
In this case $i_1 - r_1 = i_2 - r_2$

$$\text{or } i_1 - i_2 = r_1 - r_2$$

$$\text{or } \frac{i_1 - i_2}{2} = \frac{r_1 - r_2}{2}$$

$$\therefore \cos\left(\frac{i_1 - i_2}{2}\right) = \cos\left(\frac{r_1 - r_2}{2}\right)$$

$$\text{or } f = 1\,.$$

$\therefore f$ is maximum when $i_1 = i_2$.

So the deviation is minimum when $i_1 = i_2$.

Second Proof

Let us think of the standard arrangement for studying the deviation through a prism with the help of a spectrometer, where by rotating the prism table and thus changing the angle of incidence we arrive at the position of minimum deviation. The experiment confirms that the deviation is minimum and the position is unique. Now, let it happen for a particular pair of values of i_1 and i_2. We know that the path of light is reversible. So for angle of incidence i_2, the angle of emergence would be i_1 and minimum deviation will take place at the same position for the same pair of values of i_1 and i_2. It means that the roles of the angles of incidence and emergence get reversed.

Thus from the above reasoning we can say that for having minimum deviation,

$$i_1 = i_2\,.$$