Fun with Differential Equations

B V Rao

You are familiar, from high school, with the trigonometric functions $\sin x$ and $\cos x$. We shall discuss these functions starting from the differential equations that describe these functions.

B V Rao was a Professor at the Indian Statistical Institute, Kolkata. Currently he is with the Chennai Mathematical Institute.

Let us first recall our intial exposure to these functions. Fix a number α with $0 < \alpha < \pi/2$. Consider a right-angle triangle with one angle α. We defined $\sin\alpha$ as the ratio of the side opposite to the angle α to that of the hypotenuse. Here, by side, we mean the length of the side. Of course, we learnt enough about 'similar triangles' to be able to prove that this prescription does not depend on the triangle we drew. Similarly, $\cos\alpha$ is the ratio of the side adjacent to α to that of the hypotenuse. Then we extended this definition to $\alpha = 0$ and $\alpha = \pi/2$ in a manner that appealed to common sense. Then proceeded to prove, again using clever constructions, the addition formulae and so on. Finally, we extended the definitions of these functions to 'obtuse' angles, etc.

Such a treatment was good enough even to show that the value of $\sin x$ is close to x for values of x close to zero. From this followed the continuity and differentiability properties of these functions. However, the deeper analytical properties had to wait for an advanced course and depended on power series.

Here is our second exposure to these functions.

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots ,$$

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} - \cdots .$$

Keywords
Differential equations, trigonometric functions, elliptic integrals.

Properties of power series helped prove continuity and

The functions we are looking for, satisfy equations, involving the functions and their derivatives.

differentiability properties of these functions. The Cauchy product of series helped us in proving the addition rules and other properties of these functions. The geometric intuition is replaced by analytical calculations. It is not clear, though true, that the right side of the second equation sums to zero when you substitute $t = \pi/2$.

Here is our third exposure to the same functions. It uses the concept of differential equations. However, the dependence is 'essentially' illusory because, once you accept that the equations have solutions, deriving their properties is rather elementary.

In the ordinary theory of equations you learnt in high school, you were looking for numbers that satisfy some equations. For example, find numbers x satisfying $x^2 - 5x + 6 = 0$. We found that the numbers 3 and 2 satisfy this equation and *no other* number satisfies it. Find numbers x such that $x^2 + x + 1 = 0$. We found that there is no real number x that satisfies this equation.

Instead of finding numbers, we now try to find functions. The functions we are looking for, satisfy equations. Now the equations involve our functions and their derivatives. Of course, they have to be satisfied for every number t in the domain of definition of the functions. If $x(t)$ is a function of t, then $x'(t)$ denotes its derivative at t. Thus x' denotes the derivative of the function x.

Here now is our problem. We are looking for two real valued functions $x(t)$ and $y(t)$ defined on the real line which are differentiable at every point and satisfy the equations and the 'initial' conditions below.

$$x'(t) = y(t), \quad y'(t) = -x(t); \qquad x(0) = 0, \quad y(0) = 1. \tag{1}$$

All of us know that $x(t) = \sin t$ and $y(t) = \cos t$ is the only solution of the above equations. We shall now try to derive properties of these functions only with the help of

(1) without looking at any explicit formulae. You might wonder why we are wasting our time when we are in a known territory. It is always important in mathematics to arrive at the same conclusion from several vantage points. In our case, as you will see later, this expertise will help us when we are in an unknown territory, when we have *no* explicit formulae for solution.

There are general theorems assuring us of solutions. What I want you to believe is that the differential equations that we are discussing possess solutions, not because you are seeing explicit sine and cosine functions (then the purpose of my discussion is lost), but because there is a general theorem assuring you of solution. Also, there is a unique solution satisfying the differential equations having the prescribed values at the prescribed point. In our case the prescribed point is $t = 0$ and the prescribed values are as in (1).

We know that the functions x and y are differentiable. Since $x' = y$ and y is differentiable, it follows that x is twice differentiable. Since $x'' = -x$, you realize that x is differentiable any number of times – a not so exciting statement in the present case.

(i) From (1) it follows that $2xx' + 2yy' \equiv 0$, so that $x^2 + y^2$ is a constant and considering its value at $t = 0$ we conclude that $x^2(t) + y^2(t) = 1$ for all values of t.

(ii) In particular both functions take their values between -1 and $+1$ and further, they cannot simultaneously assume the value zero at any point.

(iii) Moreover, the zeros of these functions are simple. This means the following. At no point t_0 do both $x(t_0)$ and $x'(t_0)$ equal zero. This is because then both x and y are zero at that point, which cannot happen by (ii). Similarly, we cannot have any point t_0 with $y(t_0) = 0 = y'(t_0)$.

Both functions take their values between –1 and +1.

(iv) But are there zeros at all? Of course $x(0) = 0$. How

Both functions have zeros and their zeros are simple.

about y, does it ever assume the value zero? Let us consider only positive values of t. Since $y(0) = 1$ and y is a continuous function, fix a number $a > 0$ so that $y(t) > 0$ for $0 \leq t \leq a$. Thus $x'(t) > 0$ for $0 \leq t \leq a$. So x is strictly increasing in this interval and since $x(0) = 0$, we conclude that $x(a) = \eta > 0$.

We claim that $y(t) = 0$ for some $t \leq a + (2/\eta)$. If it were not so, then $y(t) > 0$ in the interval $a \leq t \leq a + (2/\eta)$. Since $x' = y$, this implies that x is strictly increasing in this interval and thus $x(t) > x(a) = \eta$, for $a < t \leq a + (2/\eta)$. Moreover by the mean value theorem,

$$|y(a + (2/\eta)) - y(a)| = x(\theta) \cdot (2/\eta)$$

for some number θ with $a < \theta < a + (2/\eta)$. But, no matter what this θ is, $x(\theta) > \eta$. Thus

$$|y(a + (2/\eta)) - y(a)| > 2$$

which is impossible because all values of y are between -1 and $+1$.

Thus y must assume the value zero. So both the functions x and y have zeros.

(v) Let $\alpha > 0$ be the first zero of y. Since $y(0) = 1$, by continuity of y, there must be a first zero; that is, $y(\alpha) = 0$ but $y(t) \neq 0$ for $0 \leq t < \alpha$. Now the fact that $x^2 + y^2 = 1$ forces $x(\alpha) = \pm 1$. Since $y(0) = 1$, we note that $y(t) = x'(t) > 0$ for $0 \leq t < \alpha$. Hence $x(t)$ increases as t increases on $[0, \alpha]$. Since $x(0) = 0$, we conclude that $x(\alpha)$ cannot equal -1. Thus $x(\alpha) = 1$.

(vi) Let us define the functions

$$\widetilde{x}(t) = -y(\alpha + t); \quad \widetilde{y}(t) = x(\alpha + t); \quad t \in R.$$

Then a routine differentiation shows that these functions also satisfy the same differential equations as x and y. Further

$$\widetilde{x}(0) = -y(\alpha) = 0; \quad \widetilde{y}(0) = x(\alpha) = 1.$$

Thus these new functions not only satisfy the same differential equations as x and y did, but also satisfy the same initial conditions. By uniqueness of solutions we then conclude that these functions are same as x and y. In other words, we have

The functions are periodic with period 2π.

$$x(\alpha + t) \equiv y(t); \quad y(\alpha + t) \equiv -x(t).$$

Hence

$$x(2\alpha + t) = x(\alpha + [\alpha + t]) = y(\alpha + t) = -x(t).$$

Similarly,

$$y(2\alpha + t) = -y(t).$$

So one gets

$$x(4\alpha + t) = x(t); \quad y(4\alpha + t) = y(t).$$

Recall that a function φ on R to R is said to be periodic if there is a number $h > 0$ such that $\varphi(t + h) = \varphi(t)$ for every real number t. The smallest such number, if it exists, is called the period of the function. It is not too difficult to show that if φ is continuous and periodic and is not a constant function, then there is indeed a smallest such number h. However, we do not need this fact in our analysis.

Thus the functions x and y are periodic with period 4α. Of course, one can also deduce successively, $x(\alpha) = +1$, $x(2\alpha) = 0$, $x(3\alpha) = -1$ and $x(4\alpha) = x(0) = 0$.

(vii) We can define this number α as $\pi/2$. But I cannot do it and fool you because you already know what is π – it is the area of a disc of radius one in the plane. So let me quickly argue that $\alpha = \pi/2$. To see this, first observe that an integration by parts gives us

$$\int_0^1 \sqrt{1-u^2}\,\mathrm{d}u = \left[u\sqrt{1-u^2}\right]_0^1 - \int_0^1 \frac{-u^2}{\sqrt{1-u^2}}\,\mathrm{d}u$$
$$= \int_0^1 \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u - \int_0^1 \sqrt{1-u^2}\,\mathrm{d}u.$$

Now we look for three functions, *x*, *y*, and *z*.

Thus

$$2\int_0^1 \sqrt{1-u^2}\,\mathrm{d}u = \int_0^1 \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u.$$

If you recall the definition of area, the integral on the left side of the above is nothing but the area of a quarter of the unit disc. Thus

$$\frac{\pi}{2} = \int_0^1 \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u.$$

You must be wondering why I did all this; is it not obvious because this integral is $\sin^{-1} u$? Yes, but if I used this, then the purpose of my discussion is defeated. I did not want to depend on our knowledge of the sine function!

Since $y(0) = 1$ and $y(t) \geq 0$ on $[0, \alpha]$, we have $x'(t) = y(t) = +\sqrt{1 - x^2(t)}$. Hence

$$\alpha = \int_0^\alpha \mathrm{d}t = \int_0^\alpha \frac{x'(t)}{\sqrt{1-x^2(t)}}\,\mathrm{d}t = \int_0^1 \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u,$$

where we substituted $u = x(t)$. As observed above, this last integral equals $\pi/2$. This completes our identification of α with $\pi/2$.

One can use the uniqueness of solutions of differential equations to prove the addition formulae for $\sin(t_1 + t_2)$, etc. But instead of continuing with this thought process, let us do something more interesting.

Now we shall consider another system. Fix $0 < \kappa < 1$. I am looking for three real-valued functions $x(t)$, $y(t)$ and $z(t)$ on the real line. They should be differentiable and satisfy the differential equations and initial conditions shown below.

$$\begin{aligned} x'(t) &= y(t)z(t); \quad x(0) = 0, \\ y'(t) &= -z(t)x(t); \quad y(0) = 1, \\ z'(t) &= -\kappa^2 x(t)y(t); \quad z(0) = 1. \end{aligned} \tag{2}$$

The first thing you notice is that these equations are non-linear – products of functions appear in the equations.

As earlier, $x(t)x'(t) + y(t)y'(t) = 0$ so that $x^2 + y^2$ is a constant function and considering $t = 0$, you again deduce that this constant must be one. In particular, these functions take values between -1 and $+1$. Also when one of them assumes the value zero, the other must take the value ± 1.

We also see that $\kappa^2 x(t)x'(t)+z(t)z'(t) = 0$ and as above we get $z^2(t) = 1- \kappa^2 x^2(t)$. In particular $z^2(t)$ always lies between $1 - \kappa^2$ and 1. Since we have taken $0 < \kappa < 1$, it follows that $z^2(t)$ never assumes the value zero. Also, since $z(0) = 1$, it follows that $z(t)$ is always positive and consequently,

$$z(t) = +\sqrt{1 - \kappa^2 x^2(t)}.$$

Do the other functions assume the value zero? Of course, initial condition tells us $x(0) = 0$. So we need to investigate if $y(t)$ is ever zero. First we fix some notation. We denote the positive square root $+\sqrt{1 - \kappa^2}$ by κ'. We also denote

$$K = \int_0^1 \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1- \kappa^2 u^2)}}.$$

Is this integral finite? Yes, precisely because $\kappa < 1$. Indeed $0 \le u \le 1$ implies that

$$\sqrt{1 - \kappa^2 u^2} \ge \sqrt{1 - \kappa^2} = \kappa'.$$

Thus the above integrand is bounded above by a multiple of $1/\sqrt{1-u^2}$ which is integrable on the interval $[0, 1]$.

Now $x'(0) = y(0)z(0) = 1 > 0$ so that $x'(t) > 0$ for t near zero. Since $x(0) = 0$, we conclude that x is positive and strictly increasing near zero. As $z(t) > 0$ for all t and x is positive near zero, we conclude, from equations (2), that $y' < 0$ near zero. Consequently, y is decreasing near zero. Of course, you can arrive at this conclusion using $x^2 + y^2 = 1$, which implies (all being positive) that if x increases then y should decrease. Suppose that $y(t) > 0$ on $[0, a]$. Thus for t in this interval $x'(t) = y(t)z(t) > 0$. A substitution of $u = x(t)$ in the first integral below gives us

$$\int_0^{x(a)} \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-\kappa^2u^2)}} =$$

$$\int_0^a \frac{x'(t)\ \mathrm{d}t}{\sqrt{(1-x^2(t))(1-\kappa^2x^2(t))}} = a\ ,$$

where for the last equality we used (2), specifically that $x' = yz$ and hence x' equals the denominator. Thus, we have

$$\int_0^{x(a)} \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-\kappa^2u^2)}} = a\ . \tag{3}$$

Since, $x(a) \leq 1$, we see that $a \leq K$ – just look at the definition of K. Thus if $y(t) > 0$ on $[0, a]$, then we must have $a \leq K$. In paticular, $y(t)$ cannot be positive for all values of t in $[0, K + 1]$. Note that K is finite and y is defined for all values of t. We conclude that y must assume the value zero for some t.

Let α be the first zero of y. Since y is continuous and $y(0) \neq 0$, there is such a first value α. Then $x^2 + y^2 = 1$ tells you that $x(\alpha) = \pm 1$. But it cannot be -1 because x is increasing in this interval from its initial value zero. Thus $x(\alpha) = +1$. Equation (3) and definition of K tell us that if $\alpha < K$, then $x(\alpha) < 1$. Thus K is the first zero of y. In particular, we have

$$x(K) = 1; \quad y(K) = 0; \quad z(K) = \kappa'.$$

Let us now define new functions

$$\widetilde{x}(t) = \frac{y(K+t)}{z(K+t)}; \quad \widetilde{y}(t) = -\kappa' \frac{x(K+t)}{z(K+t)};$$
$$\widetilde{z}(t) = \kappa' \frac{1}{z(K+t)} .$$

These are meaningful because z never assumes the value zero. These functions satisfy the same differential equations as x, y, z with the same initial conditions. We can see this as follows.

All the functions between the first and last equality below are evaluated at the point $(K+t)$. Also we use equations (2).

$$\begin{aligned}\widetilde{x}'(t) = \frac{y'z - yz'}{z^2} &= \frac{-z^2x + \kappa^2 xy^2}{z^2} \\ &= \frac{-(1-\kappa^2x^2)x + \kappa^2 x(1-x^2)}{z^2}\end{aligned}$$
$$= \frac{-x + \kappa^2 x}{z^2} = \frac{-\kappa'^2 x}{z^2} = \widetilde{y}(t)\widetilde{z}(t).$$

Similarly,

$$\widetilde{y}'(t) = -\kappa' \frac{x'z - z'x}{z^2} = -\kappa' \frac{yz^2 + \kappa^2 x^2 y}{z^2}$$
$$= -\kappa' \frac{y(1-\kappa^2x^2) + \kappa^2 x^2 y}{z^2} = -\widetilde{z}(t)\widetilde{x}(t),$$

where again, all the functions between the first and last equality are evaluated at the point $(K+t)$. The reader can verify the equation for $\widetilde{z}$.

Hence by uniqueness of solutions, these new functions must be the same as the functions x, y and z respectively. This gives us

$$x(t) = \frac{y(K+t)}{z(K+t)}; \quad y(t) = -\kappa' \frac{x(K+t)}{z(K+t)};$$
$$z(t) = \kappa' \frac{1}{z(K+t)} .$$

Properties of the functions *x* and *y* are reminiscent of the sine and cosine functions.

Thus, using the third equation in the second and then in the first, we get

$$x(K+t) = -y(t)\frac{z(K+t)}{\kappa'} = -\frac{y(t)}{z(t)} ,$$

$$y(K+t) = x(t)z(K+t) = \kappa'\frac{x(t)}{z(t)} ,$$

$$z(K+t) = \frac{\kappa'}{z(t)}.$$

By iteration, we have

$$x(2K+t) = \frac{-y(K+t)}{z(K+t)} = -x(t).$$

Similarly,

$$y(2K+t) = -y(t); \quad z(2K+t) = z(t).$$

Thus the functions x and y have period $4K$ whereas the function z has period $2K$. We also see, successively, that $x(K) = 1$, $x(2K) = 0$, $x(3K) = -1$. A similar result holds for y as well. In particular x is zero at even multiples of K, and y is zero at odd multiples of K. Also, z is maximum at even multiples of K and is minimum at odd multiples of K.

Is this not reminiscent of the sine and cosine functions? Of course, we cannot jump to the conclusion that x, y are just the sine and cosine functions; they are nearly so. Since x is strictly increasing from zero to 1 as t increases from zero to K, there is a unique increasing function $\phi : [0, K] \to [0, \pi/2]$ such that $x(t) = \sin\phi(t)$ on $[0, K]$. The relations between x, y and z then tell us that in this interval, namely $[0, K]$, we have $y(t) = \cos\phi(t)$ and $z(t) = +\sqrt{1 - \kappa^2 \sin^2\phi(t)}$.

This function ϕ is called *amplitude* – it prescribes an angle. Of course we can extend the domain of the function

ϕ in an obvious way. Since we do not need such an extension, we shall not dwell on this matter. The functions x, y and z are called the *sine-amplitude*, *cosine-amplitude* and *delta-amplitude* respectively and are usually denoted by $sn(t)$, $cn(t)$ and $dn(t)$ respectively. These are called *Jacobi Elliptic functions.*

These are called *sine-amplitude, cosine-amplitude* and *delta-amplitude*.

The true definition of the function x is revealed by (3). In fact, if you consider the area, starting from $u = 0$, under the curve given by that integrand, then it is an increasing function and the area increases from 0 to K as the upper limit of the integral increases from zero to one. And x is precisely the inverse function defined on $[0, K]$ with values in $[0, 1]$.

Before proceeding further, let us take up the limiting cases of κ . If we take $\kappa = 0$, we get the usual sine and cosine functions. Indeed the third equation reduces to $z' = 0$ so that $z(t) \equiv 1$. But then the first two equations are just what we studied earlier. Thus in this case, $sn \equiv \sin$, $cn \equiv \cos$, $\mathrm{d}n \equiv 1$ and $K = \pi/2$. Returning to the earlier discussion, we conclude the following.

The sine function is defined on $[0, \pi/2]$ as the inverse of the following function

$$\psi(u) = \int_0^u \frac{\mathrm{d}v}{\sqrt{1-v^2}} \, , \qquad 0 \le u \le 1. \tag{4}$$

This prescription can be regarded as the fourth exposure to the sine function. You may think that this method of defining the sine function is unnatural and complicated. This is simply because we were introduced to the sine function first and then we evaluated the above integral as $\sin^{-1} u$. In the present discussion, the function above appears first and then the sine function appears as its inverse.

This prescription can be regarded as the fourth exposure to the sine function.

If we take $\kappa = 1$, we see that $K = \infty$. So the period is

However the name elliptic stuck rather than hyperbolic or lemniscatic.

infinity! Indeed when $\kappa = 1$, equation (3) reduces to

$$\int_0^{x(t)} \frac{\mathrm{d}u}{(1-u^2)} = t, \quad \text{or} \quad \frac{1}{2}\log\frac{1+x(t)}{1-x(t)} = t$$

which after simplification gives us

$$x(t) = \frac{e^t - e^{-t}}{e^t + e^{-t}}, \qquad y(t) = \frac{2}{e^t + e^{-t}},$$

$$z(t) = \sqrt{1 - \kappa^2 x^2(t)} = y(t).$$

Thus in this case, we have the familiar hyperbolic functions

$$sn(t) \equiv \tanh(t); \quad cn(t) = dn(t) = \mathrm{sech}(t).$$

Why are these called elliptic functions? Because, these integrals arise while trying to calculate the arc lengths of ellipses, hyperbolas and lemniscates. However, the name elliptic stuck rather than hyperbolic or lemniscatic!

For example, consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with semi-major axis a and semi-minor axis b. Trace the arc of the ellipse in the first quadrant using $x = a\sin t$ and $y = b\cos t$ for $0 \le t \le \pi/2$ (warning: do not confuse this with usual parametrization of the ellipse). Set $\kappa^2 = 1 - \frac{b^2}{a^2}$. Then the length of of the arc is

$$\int \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}\,\mathrm{d}t = a\int \sqrt{1 - \kappa^2 \sin^2 t}\,\mathrm{d}t.$$

If you consider the lemniscate $r^2 = 2a^2\cos(2t)$, then the length of one segment of the curve is

$$\int \sqrt{r^2(t) + \dot{r}^2(t)}\,\mathrm{d}t = a\sqrt{2}\int \frac{1}{\sqrt{1-\sin^2(t)}}\,\mathrm{d}t.$$

These elliptic integrals arise when we try to understand oscillations of a pendulum. We shall not discuss this. Instead, we shall point out an exciting prospect awaiting us, if we are prepared to bring in the ever-useful complex numbers.

An exciting prospect awaits us if we are prepared to bring in the ever-useful complex numbers.

As earlier, let us fix a number $0 < \kappa < 1$. Put

$$K = \int_0^1 \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-\kappa^2u^2)}};$$
$$L = \int_1^{1/\kappa} \frac{\mathrm{d}u}{\sqrt{(u^2-1)(1-\kappa^2u^2)}}$$

Let us consider for real numbers x the integral

$$f(x) = \int_0^x \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-\kappa^2u^2)}}. \qquad (5)$$

At $x = 0$ we have $f(x) = 0$. As x increases, $f(x)$ increases reaching the value K at $x = 1$.

If you still continue further then when x exceeds 1,

$$f(x) = K + \int_1^x \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-\kappa^2u^2)}}$$
$$= K + i \int_1^x \frac{\mathrm{d}u}{\sqrt{(u^2-1)(1-\kappa^2u^2)}}.$$

Thus $f(x)$ now goes vertically, in the complex plane, from the present value $K = K + i\cdot 0$ to the value $K + i\cdot L$ as x increases from 1 to $1/\kappa$. Let us continue the integral further. When x exceeds $1/\kappa$ the second term in the radical becomes negative. Thus, now

$$f(x) = K + iL + i^2 \int_{1/\kappa}^x \frac{\mathrm{d}u}{\sqrt{(u^2-1)(\kappa^2u^2-1)}}$$

$$= K + iL - \int_{1/(\kappa x)}^1 \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-\kappa^2u^2)}},$$

Thus when both factors became negative the integrand did not return back to its original value, but to its negative because the factor $i^2 = -1$ entered.

where we changed the variable u to $1/\kappa\, u$ in the integral. Thus as x increases from $1/\kappa$ to infinity, $f(x)$ travels horizontally (backwards) in the complex plane, from $K + iL$ to iL, reaching iL when $x = \infty$.

Now let us consider negative values of x. As x decreases from zero to -1, the values of $f(x)$ run horizontally from zero to $-K$. If we still continue further down, as x decreases from -1 to $-1/\kappa$, the values of $f(x)$ travel vertically from $-K$ to $-K + iL$. If x decreases further from $-1/\kappa$ to $-\infty$, the values of $f(x)$ proceed horizontally (forward) from $-K + iL$ to iL, the same value that we reached earlier.

In summary, the function $f(x)$ defined by (5) traces the rectangle with vertices $-K$, K, $K + iL$ and $-K + iL$ in the complex plane, as x varies over the extended real line. More precisely, this is how it goes. As $x \uparrow$ from -1 to $+1$; $f(x)$ traces the lower horizontal side $[-K, +K]$. As $x \uparrow$ further from $+K$ to $+1/\kappa$; $f(x)$ traces the vertical side $[K, K + iL]$ of the rectangle. As $x \uparrow$ further from $+1/\kappa$ to ∞ $(= -\infty)$; $f(x)$ traces the upper horizontal side of the rectangle from $K + iL$ to iL. As $x \uparrow$ further from $-\infty$ $(=+\infty)$ to $-1/\kappa$, $f(x)$ continues to trace the upper horizontal side from iL to $-K+iL$. Finally, as $x \uparrow$ further from $-1/\kappa$ to -1, $f(x)$ traces the vertical side $[-K + iL, -K]$, thus completely tracing the rectangle.

You would have noticed a curious fact in the above calculation. Our integrand has two factors under the square root sign. *One after the other* they became negative, popping up i each time. Thus when both factors became negative, the integrand did not return back to its original value but returned to its negative because the factor $i^2 = -1$ entered!

The exciting prospect is: what if I consider the function defined by (5) for complex numbers z instead of only for real numbers? Let us consider the upper half plane, that is, the set of complex numbers with imaginary part greater than zero. Thus define,

$$f(z) = \int_0^z \frac{\mathrm{d}w}{\sqrt{(1-w)}\,\sqrt{(1+w)}\,\sqrt{(1-\kappa w)}\,\sqrt{(1+\kappa w)}}, \quad \mathrm{Re}(z) > 0.$$

This leads to an interesting complex function which is beyond the scope of this article. Firstly, you should note that in (5) the two factors of the denominator are in one radical sign. But now we have placed them in four radical signs! This subtle difference is to be appreciated. Secondly, you should understand that the integral is taken along a path from zero to z. The integral exists and does not depend on the path as long as it lies in the upper half plane (except for the initial point zero). Thirdly, we need to make clear as to *which* square roots are being considered. Interested readers can consult the book of Stein and Shakarchi [2].

This leads to an interesting complex function which is beyond the scope of this article.

Before concluding, let us return to equation (4). We mentioned that (4) can be made the basis for another definition of sine function. More precisely sine function is the inverse of the function defined by (4). But such a procedure gives a sine function, say $x(t)$, only on the interval $[0, \pi/2]$. How do you go further?

Well, you reflect! Since x' reaches zero when you reach $\pi/2$, you can define it on $[\pi/2,\ \pi]$ simply by $x(t) = x(\pi - t)$. This preserves differentiability; left derivative $= 0$, right derivative $= -0$. Having got $x(t)$ on $[0, \pi]$, how do you extend it further – again reflect?

If you were to blindly imitate and set $x(t) = x(2\pi - t)$, then there is a problem. Since $x'(\pi) = 1$ from left, such a definition as the proposed one makes $x'(\pi) = -1$ from right, violating differentiability. Now the saving grace is $x(\pi) = 0$ and taking a negative would preserve continuity as well as differentiability. Thus we define $x(t) = -x(2\pi - t)$ for $\pi \le t \le 2\pi$. This will then be differentiable on $[0, 2\pi]$. It is now clear how to define for all real numbers t.

Suggested Reading

[1] A I Markushevich, ***The Remarkable Sine Functions***, **Elsevier, New York, 1966.**

[2] **Elias M Stein and Rami Shakarchi, *Complex Analysis*, Princeton, Univ. Press, 2003.**

[3] **F G Tricomi, *Differential Equations*, Hafner Publishing Company, New York, 1961.**

Address for Correspondence
B V Rao
Chennai Mathematical Institute
Plot H1, SIPCOT IT Park
Siruseri, Padur Post
Chennai 603 103, TN, India.
Email:
bhamidivrao@gmail.com