Remembering Shreeram S Abhyankar

Sudhir R Ghorpade

Shreeram Shankar Abhyankar, an extraordinary mathematician and educator, passed away on November 2, 2012 at the age of 82. He remained active in research as well as teaching and was constantly engaged in "doing mathematics" almost till his last breath. This, and the accompanying articles, are meant as a tribute to Prof. Abhyankar and an attempt to provide some glimpses of the person and his work.

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1. The Beginnings

Shreeram Abhyankar was born on July 22, 1930 at Ujjain in central India. His father, Shankar Keshav Abhyankar, was a professor of mathematics based in Gwalior and he worked in Ujjain during 1928–1932, while his mother Uma Abhyankar (nee Tamhankar) was originally from Burhanpur. Shreeram was a prodigious child with an exceptional aptitude for mathematics. After completing his schooling in Gwalior, he came to Bombay and studied at what was then called the Royal Institute of Science. Around this time, he began visiting the newly founded Tata Institute of Fundamental Research (TIFR) and attending some lecture courses there, most notably the one by Marshall Stone. Even though he was intensely interested in mathematics, some doubts still lingered in his mind and at one time, young Shreeram felt that he should major in physics. What brought about the change is a story that I have always found awe-inspiring and goes as follows. When asked by D. D. Kosambi, one of the two professors (besides F. W. Levi) at the TIFR then, the reasons for majoring in physics in spite of being more interested in mathematics, Shreeram Abhyankar mentioned "I don't know what I should do if one day I can no longer do mathematics". To this, Kosambi replied, "Then you should kill yourself." At that moment, all the doubts in Abhyankar's mind melted away and, as we now know, he has never had a reason to kill himself! The head of

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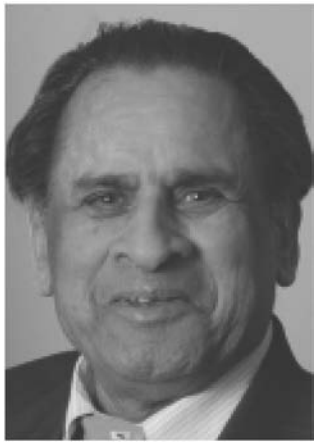

Shreeram S Abhyankar

Keywords
Abhyankar, algebraic geometry, resolution of singularities, affine geometry, Young tableaux and determinantal varieties, Galois theory and group theory, Jacobian problem and dicritical divisors.

the Department of Mathematics at the Royal Institute of Science at that time was Pesi Masani, a 1946 Harvard PhD with Garrett Birkhoff and a close associate of Norbert Wiener. With some encouragement from him, Abhyankar could enroll in the PhD programme at Harvard University on the eastern coast of USA. Thus, after obtaining his BSc in Mathematics from Bombay University in 1951, Shreeram Abhyankar embarked on his first overseas voyage to the United States.

2. Inception and Evolution of a Research Career

Owing to an illness on the boat while travelling to the USA, Abhyankar was detained in England for about two months and reached Harvard later than scheduled. As Sathaye has mentioned (see Appendix 1), Abhyankar met, on his first day in the US his future PhD supervisor Oscar Zariski, who was already a legendary figure in algebraic geometry. Shreeram Abhyankar received the MA and PhD degrees from Harvard University in 1952 and 1955 respectively. His thesis work was a major breakthrough where he succeeded in settling the problem of resolution of singularities of algebraic surfaces in prime characteristic. Invitations to several places followed and in the subsequent years, he held regular and visiting positions at some of the leading universities worldwide including Columbia, Cornell, Johns Hopkins, Harvard, Princeton and Yale in USA, Erlangen and Münster in Germany, Leiden in Holland, Angers, Nice, Paris, Saint-Cloud and Strasbourg in France, and Kyoto in Japan. In 1963 he moved to Purdue University, West Lafayette, Indiana, USA, and since 1967 he was the Marshall Distinguished Professor of Mathematics at Purdue. Moreover, since 1987–1988, he was also made a Professor in the Departments of Industrial Engineering and Computer Science at Purdue.

Throughout his life, Shreeram Abhyankar retained close ties to the country of his birth and was deeply concerned about the development of mathematics in India. He made numerous visits to academic institutions in India including TIFR, IIT Bombay, MatScience, and in fact, spent several years away from Purdue

while he worked as a Professor (and for some years as the Head of Department of Mathematics) at the University of Pune. Moreover, he founded a research institution, named Bhaskaracharya Pratishthana, at Pune in 1976. His inspiring seminars and eminence in mathematics often attracted many students and several of them went on to do PhD with him.

3. Research Contributions

Shreeram Abhyankar has made numerous important contributions to many areas of mathematics, especially algebraic geometry, commutative algebra, theory of functions of several complex variables, invariant theory, and combinatorics. He has authored close to 200 research papers published in some of the leading international journals. He is also the author of about a dozen books and research monographs. Like the great David Hilbert, Abhyankar's work can be roughly divided in fairly distinct phases during which he focused mainly on one broad topic and made substantive contributions. (See also the write-up by Balwant Singh in Appendix 2.) A brief outline is given below.

Shreeram Abhyankar has made numerous important contributions to many areas of mathematics, especially algebraic geometry, commutative algebra, theory of functions of several complex variables, invariant theory, and combinatorics.

Phase I: Resolution of Singularities. This phase began, as noted earlier, with the path-breaking PhD thesis work of Shreeram Abhyankar, and was predominant mainly during 1954–1969. The topic always remained close to his heart and he would return to it from time to time, most notably, in the early 1980's when he worked on canonical desingularisation. See the accompanying articles of Balwant Singh and Dale Cutkosky (Appendix 2 and 3) for more on this topic. We also refer to the brief survey by Mulay in [7] and a riveting account, intermingled with personalised history, by Abhyankar himself in his Bulletin article [6].

Like the great David Hilbert, Abhyankar's work can be roughly divided in fairly distinct phases during which he focused mainly on one broad topic and made substantive contributions.

Phase II: Affine Geometry. The study of affine algebraic curves and surfaces together with their embeddings and automorphisms formed the focal theme of Abhyankar's researches beginning around 1970 and lasting for about a decade. Through his explicit and algorithmic methods, rooted in high school algebra, Abhyankar achieved remarkable successes that greatly advanced the research in affine algebraic geometry.

Some brilliant collaborations with his PhD students, especially T. T. Moh and Avinash Sathaye, ensued during this period. Abhyankar also popularised a 1939 conjecture of Keller, now known as the Jacobian problem, that still remains open and has enticed countless mathematicians over the years. The reminiscences by Avinash Sathaye, Balwant Singh and Rajendra Gurjar (see Appendices 1, 2 and 4) shed more light on this topic and the contributions of Abhyankar in it. Further, one may refer to the relevant parts of the Ford and Chauvenet award winning article [1] and the Kyoto paper [2] of Abhyankar for more on this topic.

Phase III: Young Tableaux and Determinantal Varieties. Around 1982, motivated by some questions concerning singularities of Schubert varieties in flag manifolds, Abhyankar became interested in determinantal varieties and was fascinated by the straightening law of Doubilet–Rota–Stein. He forayed into the combinatorics of Young tableaux and by a remarkable *tour de force*, managed to discover intricate formulas for enumerating various classes of tableaux and the interrelationships among them. These could be applied to obtain Hilbert functions of determinantal varieties and to derive some geometric properties. It was during this period that I met him and became enamoured of his mathematics. Abhyankar's work on this topic is mainly found in his research monograph [3] of about 500 pages that, incidentally, has no Greek symbols! For a gentler introduction, one may refer to my survey article in [7]. Although Abhyankar's interest in combinatorics had reached a peak in the mid-80's to the extent that he was once willing to denounce being an algebraic geometer, by the late-80's, he was persuaded by engineers and computer scientists to return to algebraic geometry and to teach them his algorithmic approach to it. This resulted in profitable interactions and was partly responsible for his book *Algebraic Geometry for Scientists and Engineers* [4] that went on to become an AMS bestseller. And then there were the letters from J.-P. Serre.

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Phase IV: Galois Theory and Group Theory. In a series of letters in 1988, Serre posed specific questions to Abhyankar related to his 1957 paper on covering of algebraic curves. This led

to a revival of interest by Abhyankar in his own conjectures concerning Galois groups of unramified coverings of the affine line in characteristic $p \neq 0$. By this time, group theory had progressed a great deal and Abhyankar, already in his sixties, became a student again and began a massive study of the significant advances in group theory and applied them effectively to questions in Galois theory. During 1990–2005, Abhyankar wrote numerous papers giving "nice equations for nice groups". Meanwhile, Abhyankar's conjectures from his 1957 paper were settled using rather abstract methods, by M. Raynaud and D. Harbater (for which they received the Cole Prize from the American Mathematical Society). For an introduction to Abhyankar's work during this phase, it may be best to refer to his two Bulletin articles [5] and [6].

Phase V: Jacobian Problem and Dicritical Divisors. Soon after the publication in 2006 of his book *Lectures on Algebra* I that has more than 700 pages, Abhyankar returned to the Jacobian problem, which was mentioned earlier. He began by publishing his "thoughts" in a four part paper of more than 300 pages in the *Journal of Algebra* in 2008. Around this time, he encountered the topological notion of dicritical divisors. He quickly understood its significance, but was not satisfied until he could understand them algebraically in his own way. In a series of remarkable papers, he algebracised the theory of dicritical divisors thereby making it valid for nonzero characteristic as well as mixed characteristic, and studied its connections with the Jacobian problem. As he liked to put it, using dicritical divisors, one sees that intrinsically hidden inside the belly of a bivariate polynomial there live a finite number of univariate polynomials. Sathaye had once described this work of Abhyankar's as one of his best thus far. One is further impressed when one considers that it was done when Abhyankar was around 80 years of age!

Abhyankar's work on dicritical divisors had been described as one of his best thus far! And it was done when Abhyankar was around 80 years old!

4. Honours and Accolades

Quite naturally, Shreeram Abhyankar received numerous honours and awards during his lifetime. This includes McCoy Prize from

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Purdue, Lester Ford Prize and the Chauvenet Prize from the Mathematical Association of America, a Medal of Honour from the University of Valliadolid, Spain, as well as the University of Brasilia, Brazil, and the honorary title of *Vidnyan Sanstha Ratna* from the Institute of Science, Mumbai. He received a honorary doctorate from the University of Angers, France in 1998. On this occasion, Fields medalist Heisuke Hironaka wrote: "*Your long and powerful works deserve far more than the honorary doctorate you are receiving. Even so, I am happy to hear the good news. Your originality has been a gold mine for many other algebraic geometers, including myself. Now the mined gold is receiving rays of sunlight, facets after facets*."

Abhyankar was elected as a Fellow of the Indian National Science Academy in 1987 and the Indian Academy of Sciences in 1988. Most recently, he was among the inaugural batch of Fellows of the American Mathematical Society (AMS) announced by the AMS on November 1, 2012. He has guided about 30 PhD students and has inspired many more at different stages of their education and research career. International conferences in algebra and algebraic geometry in honour of Shreeram Abhyankar were held at Purdue in 1990, 2000, 2010 and 2012 around the month of July and also at Pune, India, in December 2010.

5. Epilogue

Professor Abhyankar is survived by his wife Yvonne, a remarkable person herself and who had been a constant companion and a source of strength to him since their marriage in 1958, son Hari (a 1999 PhD in Operations Management from MIT), daughter Kashi (a 2001 PhD in Mathematics from Berkeley), and four granddaughters Maya, Kira, Kaia and Ela.

Shreeram Abhyankar's influence on many areas of mathematics, especially algebraic geometry, through his outstanding research and on numerous students, colleagues, and admirers the world over through his inspiring lectures and also his books and articles, shall remain for years to come. Those of us who have had the pleasure and privilege of knowing him and learning something

about mathematics and life from him will always cherish the fond memories of our association.

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Appendix 1
Shreeram Shankar Abhyankar: Appreciation of my Guru

I met Abhyankar when I was a third year college student, while he was visiting Pune in Summer. He had made an open invitation to anyone interested in mathematics and our mathematics Professor took me to meet him in response. My introductory meeting lasted about three hours, to be repeated over several days! It is hard to summarise what we talked about. He would certainly answer any questions about mathematics that I asked, sometimes diverting them to more interesting topics. Often, he would talk about Marathi and Sanskrit literature, philosophy, memories of learning in childhood and so on. Very rarely did he give a formal lecture, in these private chats. He was always patient in explaining the same thing over and over again.

I continued to visit him whenever he would return to Pune and finally in 1969, I moved to Purdue University after he suggested that would make it easier to continue my studies. I finished my doctorate in 1973 and have been deriving my inspiration to do mathematics by observing his work and listening to him.

Abhyankar had a unique perspective of mathematics. He often rebeled against "fancy mathematics", demanding that all theorems should have detailed concrete proofs. He also believed that papers should spell out all the necessary details and he practiced this rigorously. As a result, several of his papers are difficult to read, because you have to keep your concentration on every little detail that he has laid down. He would often say that the proofs should be so logical that a computer should be able to verify them!

He also had a sense of poetry and rhythm in his papers. He would create multiple subsections with matching words and equal number of subitems, so that the paper had a natural symmetry. Sometimes, he would spend enormous amount of time to create such intricate structures. He then would proceed to prove the main theorem by pulling together his several lemmas into a proof of the type: "The result follows by items $a, b, c, \ldots, z$".

He was also a master of proof by mathematical induction. A good induction proof needs a clear understanding of the main points and an intellectual capacity to analyse the changes as you move through different cases.

During his long mathematical career of more than 57 years, he went through several distinct periods of concentration on specific topics. When he was concentrating on a specific topic, he would be totally immersed in it, reading what he could find, asking everybody about it, and even consciously seeking the experts in the field. At the end, he would become a master of the subject himself.

I was his student when he was concentrated on affine geometry. He had just finished his monumental work in resolution of singularities and had come to the conclusion that he needed something appealing to young new students. The subject of resolution, while at his heart, required years of preparation and did not connect that well with freshmen or high school students. By his personal experience, he knew that love of mathematics is best developed early — he was younger than 10 when he discovered its beauty.

So he thought of interesting problems about the simplest mathematical structure, the polynomials. This is something one learns in middle school and usually stays in one's mind as a boring skill! He was determined to change that.

He invented the question which has now become famous by the title "Abhyankar–Moh Epimorphism Theorem". In a modern textbook, it would be stated in fancy language as follows:

Suppose that $f \in k[X, Y]$ — a polynomial in two variables such that f is biregular to a line, then is f a generator of the polynomial ring?

While precise, this statement needs lots of explanation. Abhyankar reformulated it so that even a middle schooler can understand and think about it:

Suppose $p(t) = t^n + p_1 t^{n-1} + \cdots + p_n$ and $q(t) = t^m + q_1 t^{m-1} + \cdots + q_m$ are polynomials so that t can be written as a polynomial in $p(t)$ and $q(t)$. Is it true that n divides m or m divides n?

Students are familiar with linear change of variables. Building on the concept, Abhyankar described a polynomial $f(X, Y)$ to be a "variable" if there is a polynomial $g(X, Y)$ such that every polynomial in X, Y can be written as a polynomial in X, Y. In standard notation, this means $k[X, Y] = k[f, g]$.

Then the central question raised by Abhyankar was, how can you tell if a given $f(X, Y)$ is a variable? The Epimorphism Theorem gives a sufficient condition that there are polynomials p, q as described above, so that $f(p(t), q(t)) = 0$. (This is actually true only when you are working in characteristic zero, but that means it is true in the usual real or complex numbers.)

The corresponding three (or higher) dimensional question is still unresolved, but has been a key feature of numerous research papers since.

Another intriguing question is how to tell if a given pair of polynomials f, g form a pair of variables, i.e. $k[X, Y] = k[f, g]$. An answer in the form of the famous "Automorphism Theorem" is that we should be able to transform the given f, g into X, Y by a sequence of standard transformations where we hold one of them fixed and add a polynomial expression in it to the other. For example, replace f, g by $f, g + 2f - f^3$.

While this is a valid criterion, it is not that satisfactory, since we do not know the result until all the steps are carried out. So it is worth seeking other useful criteria.

Abhyankar popularised another striking question known as the Jacobian Problem which asks:

Suppose that the Jacobian of f, g written as $J(f, g) = f_X g_Y - f_Y g_X$ is a nonzero constant. Then is it true (in characteristic zero, or simply in complex numbers) that f, g is a pair of variables?

A calculus student knows and can understand this condition. This simple sounding problem has a long history. There are several published incorrect proofs and new ones are being produced with a predictable regularity. Abhyankar himself was instrumental in pointing out the flaws in many of these "proofs" and has some of the best results obtained so far (at least in the two variable case). The problem naturally extends to any number of variables and is an active area of research.

Abhyankar has also led in and inspired a lot of research in the problems of Galois Theory (especially over function fields in positive characteristic). Abhyankar used to fondly recall how his pathbreaking papers on fundamental groups (from 1950s) were born out of a flash of intuition during intense concentration. He felt that he had experienced the yogic experience of Samadhi at that time and he had practically decided to become a Yogi instead! Fortunately for the mathematical world, some accidental events in his life at the time brought him back to the materialworld. Yet, till the end, he always considered his mathematics as applied Yoga!

Abhyankar was convinced that mathematics is a panacea. He used to tell how, as a young child he was weak and suffered from many ailments. This continued until he discovered mathematics. Once he started reading mathematics, he did not get sick again. Even when sick, he could push aside the pain and get well by immersing in mathematics. Perhaps, his sudden death while sitting at his desk, working on mathematics, is a testimonial to his theory of mathematics over matter!

Avinash Sathaye

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Appendix 2
A Note on Abhyankar and His Work

Abhyankar's style of work, according to my observation, was to work thoroughly in one area of algebraic geometry for a few years, making substantial and deep contributions to it, and then move on to a different area or return to one of his earlier favourites. Some specific areas encompassed in his vast research work are resolution of singularities, tame coverings and algebraic fundamental groups, affine geometry, enumerative combinatorics of Young tableaux and Galois groups and equations. I will talk about only two of these, where I have some firsthand knowledge, namely resolution of singularities and affine geometry. In the 1940's Zariski had obtained a rigorous proof of resolution of singularities of surfaces and threefolds in characteristic zero. The case of positive characteristic for surfaces was done by Abhyankar in his PhD thesis in 1956. An extension of the result to threefolds required, as a first step, the resolution of singularities of an embedded surface. This Abhyankar investigated over several years, developing in the process highly intricate where I have some firsthand knowledge, namely resolution of singularities and affine geometry.

In the 1940's Zariski had obtained a rigorous proof of resolution of singularities of surfaces and

threefolds in characteristic zero. The case of positive characteristic for surfaces was done by Abhyankar in his PhD thesis in 1956. An extension of the result to threefolds required, as a first step, the resolution of singularities of an embedded surface. This Abhyankar investigated over several years, developing in the process highly intricate and powerful algorithms in positive characteristic. As the material grew in size, he found it necessary to write up and update his various results in a book *Resolutions of Singularities of Embedded Algebraic Surfaces*, which appeared in 1966 and which culminated in a proof of resolution of singularities of threefolds in characteristics other than 2, 3, 5. He once told me that he had about 300 pages of handwritten notes which take care also of characteristics 2, 3, 5, but that he did not have the energy to translate these into a readable exposition. Apart from positive characteristic, Abhyankar also solved the equally difficult problem for the arithmetic case, i.e. for surfaces over the ring of integers. For a long time the only significant contributions in positive characteristic or in the arithmetic case were those due to Abhyankar.

In affine geometry, the themes of Abhyankar's work were embeddings and automorphisms. Two well-known terms here are the Epimorphism Theorem and the Jacobian Conjecture. In fact, Abhyankar's focus was on the Jacobian Conjecture, and Epimorphism Theorem was just the outcome of his first attempt at solving the Jacobian Conjecture. This was in the early 1970's. Soon after, Abhyankar moved to other areas but then returned to the Jacobian Conjecture in about the year 2002. Then this remained the area of his work in the last decade of his life. Abhyankar was very fond of *algebraicising* results from other areas, notably analysis and topology, if he thought they were relevant to his current interest. He often succeeded in doing this, and this was also one of his strengths. It is in this spirit that he developed the algebraic theory of dicritical divisors in the last few years, keeping in view their possible application to a solution of the Jacobian Conjecture.

I must single out two of his publications for special mention: His paper *On the valuations centered in a local domain* (1956) and his Princeton monograph *Ramification Theoretic Methods in Algebraic Geometry* (1959). These are repertories which continue even now to yield new techniques and insights.

I discovered early on in my association with Abhyankar that while you can learn mathematics from a book you can almost never get from it the insight which listening to his lectures provided. I learned much mathematics from his lectures and my private discussions with him.

My relationship with Abhyankar went beyond mathematics. Our families have had a close relationship for over four decades. In my young days I took pride in my knowledge of Hindu mythology. This pride evaporated quickly after I heard Abhyankar expound on the theme for the first time. He must have noticed my interest because soon mythology became as much a part of

our conversations as mathematics, if not more. Mostly this was a one way street with me learning much mythology and mathematics from him. During a talk by Abhyankar at Purdue on "Relationships in Mahabharata" the audience was awestruck by his unmistakable and firm grasp on the complicated web of inter-relationships among numerous characters in the epic. I must say that over the years I developed a distinct feeling that some of his actions were guided by what he believed his mythological hero did or would do in a similar situation.

Balwant Singh

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Appendix 3
An Homage to Professor Abhyankar

It was with great sorrow that I heard of the passing of Prof. Abhyankar. He was a great man. I admired and liked him, both as a mathematician and as a man. I first met him in West Lafayette in 1988, when my wife, Hema Srinivasan, and I were visiting Purdue University for a year. We frequently visited his home, and have continued to have a close contact with him since then. Always he was surrounded by his students and co-workers. His wife, Yvonne, somehow managed all of this, making everyone welcome. At that time, their son Hari was an undergraduate at Purdue and their daughter Kashi was still in High School.

Professor Abhyankar was a charismatic man, and an excellent speaker, who could mesmerise an audience. He had a special way of talking with people about mathematics. He would insist that people explain what they were doing in elementary terms. Often when you began talking with him you realised how poor your understanding was, but at the end of the conversation you had a much deeper knowledge. Throughout his mathematical life, he took polynomials and power series, and related concepts such as determinants and discriminants as the focus of his interest, only considering the most fundamental and important problems. He liked to see things in the simplest way possible, without affectation. I have always admired his strength, being willing to stand alone if necessary, following what he believed in.

I have spent many hours closely studying Professor Abhyankar's wonderful papers on resolution, valuations and ramification. His papers are written with remarkable care. It takes

tremendous effort to read them. He created his own language to explain his mathematics. Everything is stated in pure algebra, with no recourse to possibly misleading geometric intuition. Often a key point in the argument is a clever algebraic manipulation. Considering the incredible complexity of his proofs, it is amazing how carefully written and precise they are.

I am very glad that I was able to attend his 82nd Birthday conference this summer at Purdue University. The conference was a tribute to his rich life, celebrated by his many students, collaborators and colleagues.

Steven Dale Cutkosky

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Appendix 4
Professor Shreeram Shankar Abhyankar: Some Reminiscences

I first met Prof. Shreeram Abhyankar sometime in 1970 when I was still an undergraduate in Pune. Avinash Sathaye, who was then a PhD student of Abhyankar, took me to Abhyankar's house. For the next few years I used to visit him when he came to Pune. During these visits he had recommended some books in mathematics for me to read. Among them I remember the following classics: Kamke's *Set Theory*, Knopp's two-volume *Theory of Functions*, Zariski-Samuel's two-volume *Commutative Algebra*, Ford's *Automorphic Functions*, Siegel's three-volume *Complex Function Theory*, Goursat's *Complex Analysis*, ...

During my student years, and later as a more mature researcher, I managed to read all these books. The knowledge acquired from these books has helped me greatly over the years and made me a more confident researcher. I wish I had read more of the books suggested by Abhyankar!

Professor Abhyankar helped me to go to Purdue University for PhD. I had the honour of associating with him in one research project. In connection with the famous Jacobian Problem, I found a formula due to Lagrange for expressing the inverse of a power series in one variable which has a non-vanishing determinant at the origin. In Goursat's book there was a partial generalisation of this to two variables. By some heuristic argument I found a complete generalisation to arbitrary number of variables using the argument in Goursat which used

complex integration. When I showed this to Abhyankar he first observed that I was expressing the formula in a somewhat complicated way. He first simplified the expression and then gave a purely algebraic proof of it! Several mathematicians, including Bass–Connell–Wright, have tried to use this formula for solving the Jacobian Problem. From this proof, and from my study of some of Abhyankar's work in later years, I consider him as one of the best commutative algebraists in the world in the last 60 years.

Before settling down in Purdue University in 1963 he had worked in major universities like Columbia, Cornell, Princeton, Johns Hopkins, Harvard, . . .

Professor Abhyankar has many outstanding results to his credit. There is still no improvement, or simplification, of his work about resolution of singularities of surfaces in 1956, and threefolds in 1965 in characteristic $p > 0$. Some of the other works for which he will always be remembered are: Uniqueness of embeddings of an affine line in the affine plane in characteristic 0, Galois groups of coverings of the affine line in characteristic $p > 0$, multiply-transitive Galois covers, Jacobian Problem in two dimension. In my opinion he went much deeper than anyone else in trying to solve the Jacobian Problem.

Professor Abhyankar authored many books: *Resolution of Singularities of Embedded Algebraic Surfaces, Ramification Theoretic Methods in Algebraic Geometry, Local Analytic Geometry, Algebraic Space Curves, Algebraic Geometry for Scientists and Engineers, Expansion Techniques in Algebraic Geometry, Enumerative Combinatorics of Young Tableaux, Weighted Expansions for Canonical Desingularization*, a recent textbook on *Algebra*, . . . These will be valuable for years to come.

The total body of his work (research papers, books, a very large number of conference lectures around the world, . . .) is truly staggering. It shows his tremendous hard work, technical abilities and total passion for mathematics. This in itself is greatly inspiring, but he will always be remembered for his indomitable spirit, a strong patriotic feeling towards his Indian roots, his confidence in his knowledge and a just pride in knowing the value, and place, of his research work in algebraic geometry. It can be said that his interests lay more in classical mathematics. Perhaps I am wrong, but I think he did not even once use tensor product in his research!

One of his motto was never to use a result whose proof he had not read. He broke this rule the first time when he used the classification of finite simple groups.

I met him in August during a conference on Topology of Algebraic Varieties in Montreal and we had interesting discussions about the latest paper he was writing with Artal Bartolo, hoping to make Zariski's work on complete ideals in two-dimensional regular local rings more

understandable. Even at 82, he was in great spirit. In the conference he gave a talk on some nice work he had done a few years ago using classical Knot Theoretic work of Zariski related to singularities of plane algebraic curves. This work of Abhyankar is also related to his work on some version of Hilbert's 13th Problem. He remained active and thought about mathematics almost till his last breath.

Professor Abhyankar guided more than twenty five students for PhD. More than ten of them are Indian; in fact Prof. Abhyankar took many Indian students to Purdue and guided them for PhD. He made a large number of visits to India and inspired many young mathematicians. I count myself lucky to be one of these people. In recent years, during his visits he used to lecture to many students from Pune/Mumbai on diverse topics in algebraic geometry, sometimes for hours. My son Sudarshan was one of these students; this, forty years after I had first started learning from him!

Although he lived in US for more than sixty years he contributed to Indian mathematics in many ways. His founding of the Bhaskaracharya Pratishthana in Pune is only one of these important contributions.

Professor Abhyankar received many prestigious awards and fellowships in his long and distinguished career. Some of the prestigious awards like, Sloan Fellowship, Ford Prize, Chauvenet Award, . . . are the proof of his outstanding works. He was Marshall Distinguished Professor of Mathematics in Purdue University from 1967. For the past twenty-five years he was also a professor in the departments of Industrial Engineering and Computer Science in Purdue University, in recognition of his interdisciplinary work.

Professor Abhyankar will always remain alive in our memories.

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