Marin Mersenne, 1588–1648

Shailesh A Shirali

Marin Mersenne was many things: scientist, mathematician and writer. He did original work on acoustics and on prime numbers; today his name is linked to a family of primes called 'Mersenne primes'. But his greatest contribution was his work in propagating a culture of scientific inquiry in Europe – a culture which emphasized communication and dissemination and learning from one's peers. The work done on the cycloid illustrates this theme beautifully. This article looks at some work for which Mersenne is best known.

Shailesh Shirali is Director of Sahyadri School (KFI), Pune, and also Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been in the field of mathematics education for three decades, and has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books addressed to high school students, and serves as an editor for *Resonance* and for *At Right Angles*. He is engaged in many outreach projects in teacher education.

Introduction

Is it possible for any one person to serve as a 'human equivalent of the World Wide Web'? And could this have been possible in the post-Renaissance world four centuries back, when letters could take weeks to travel from one point to another? In this article we document the story of one individual who appears to have done just this: Father Marin Mersenne (1588–1648), who lived in France during an era of intellectual giants, from his country and elsewhere: René Descartes (1596–1650), Étienne Pascal (1588–1651) and his son Blaise Pascal (1623–1662), Pierre de Fermat (1601–1665), Gilles Roberval (1602–1675), Girard Desargues (1591–1661), Galileo Galilei (1564–1642), Johannes Kepler (1571–1630), Evangelista Torricelli (1608–1647) and others, with whose names are linked so many great discoveries and inventions in mathematics and mechanics: the marriage of

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algebra and geometry (Descartes), the birth of probability theory (Fermat and Pascal), results in number theory (Fermat), results in geometry and hydrostatics (Pascal), advances in projective geometry (Pascal and Desargues). Against this backdrop, the work of Mersenne in sound and number theory may seem modest. But we shall show that Mersenne's real contribution lay in quite another area. Living in an era which was enormously productive for mathematics and mechanics – the decades preceding Newton and Leibnitz, and the decades during which the seeds of the calculus were being sown – his role was absolutely central.

Figure 1. Father Marin Mersenne in his cell. Source: [2]

Early Years

Marin Mersenne was born in September 1588, in a town called Oizé. His parents were of modest means, but they took pains to attend to his education, sending him to a Jesuit School – indeed, the same school attended by Descartes some years earlier. He was expected to join the Church (we find this theme occurring time and again in such biographies!), but his eagerness to study further had a decisive role in shaping his future. On his way to Paris for further studies in philosophy and theology, he stayed at a convent of the Order of the Minims, and this brief contact touched him deeply. He completed his studies in the Sorbonne in 1611, and promptly joined the Order of the Minims, in Paris.

Mersenne attended a Jesuit school which was also the school attended by Descartes some years earlier.

The Minims

The Order had been set up two centuries earlier. They regarded themselves as the 'least' of all the religions; hence their name, 'minims'. They gave a high place to study and scholarship. It must have been these aspects of the Order that so attracted Mersenne. It was their practice to be clothed in a habit made of coarse wool,

The 'Minims' (to which Mersenne belonged) regarded themselves as the 'least' of all the religions.

and this is what we see in all the surviving portraits of Mersenne.

It was at the monasteries of the Order that Mersenne's style of work and communication with fellow scholars began to take shape.

By 1614 Mersenne was teaching philosophy and theology in the monasteries of the Order. Around this time he discovered and explored the curve known as the *cycloid*. (We shall say more about this later in the article.) It was during this period that his characteristic style of work began to take shape: the way he maintained links with scholars and exchanged ideas with them. Mersenne stayed at the Place du Royale monastery in Paris from the second half of this decade to the end of his life – a stay of over thirty years, supported for the most part by the Church. As his biography states, " . . . The Minims realised that the biggest service he could give was through his books and they never asked any more of him" (see [5]).

By the early 1630s Mersenne was convinced that Galileo's ideas needed to be better known in Europe.

Transformation

A remarkable transformation happened in Mersenne's life over the 1620s. His initial writings were mostly on religious topics, and he would have been regarded as "pro Aristotle" and "anti Galileo". But in the late 1620s he studied the criticisms made against Galileo very closely, and by the early 1630s he was convinced that Galileo's ideas needed to be propagated. By this time he had also decided for himself that alongside religious studies he would devote his time to science and mathematics. This interest in combination with his meetings with scholars soon gave rise to an extraordinary and unprecedented tradition, in which he began to keep contacts with a number of scientists and mathematicians (including some whom he never met): Descartes, Fermat, Étienne Pascal and Blaise Pascal, Roberval, and many others. He set up meetings with them in which they

His informal meeting space began to be known as Mersenne's Academy.

would discuss their work. This informal academy began to be known as *Académie Mersenne*. At one such meeting Mersenne persuaded Roberval to work on the cycloid, and this brought forth rich dividends.

Mersenne discovered the law relating pitch of a plucked string to the tension, length and thickness of the string.

Mersenne's Work in Music and Sound

Meanwhile Mersenne continued his own research. Here is an example of one of his discoveries, in sound (he had always had a deep interest in acoustics and music). The Pythagoreans had discovered, early on, that as one varies the length of a vibrating string, the pitch of the resulting note varies in inverse proportion to the length. When one halves the length of the string, the pitch doubles; we ascend through one octave. This simple arithmetical relationship along with the fact that pleasing musical harmonies correspond to 'simple ratios' like 2 : 1, 3 : 2, 5 : 3 and so on, had deep implications for the way the Pythagoreans viewed the world, for it suggested that the world had to be constructed on mathematical principles. But pitch must depend on other factors as well; one only has to look at the different strings of a guitar or sitar to see this. Mersenne was the first person to investigate this matter in a systematic and scientific manner, and he discovered the following beautiful law relating pitch to the tension, the length and the thickness of the vibrating string:

His interest in music also led him to study the topic of permutations and combinations.

$$\text{pitch of sound} \propto \frac{\sqrt{\text{tension}}}{\text{length} \times (\text{diameter})^2}.$$

Mersenne was not content with the purely empirical discovery and strove hard to establish it using the known principles of mechanics. All his thoughts and discoveries on this topic are contained in a book he published in 1630, *Harmonie Universelle*.

Figure 2. Cover page of Mersenne's *Harmonie Universelle.*
Source: [4]

His interest in music and the theory of musical harmony

To inform Mersenne of a discovery meant to publish it through the whole of Europe.

also led him to explore a topic which in those days would not have seemed to have anything to do with mathematics: permutations and combinations!

The Human Web

Through the 1630s he continued his work in encouraging others and making their results known to others. It was said of him: *To inform Mersenne of a discovery meant to publish it through the whole of Europe.* For example, following a visit to Italy in the early 1640s he learnt of Torricelli's work on the barometer and immediately set about publicizing it, while at the same time conducting many experiments of his own in the same field.

Even Galileo benefited from his zeal. Characteristically, Mersenne set about verifying all of Galileo's findings when he heard about them; for example, the law governing the speed of a falling body – the formula every school student remembers as $s = \frac{1}{2}gt^2$. In his usual way he then made Galileo's work known to others. Another scientist who benefited greatly from the kindness shown by him was Christiaan Huygens (1629–1695); he must have been just a teenager at the time. Mersenne posed a challenge to him: to determine the equation of the catenary. But this challenge proved too difficult for the young Huygens.

Another scientist who benefited from the kindness shown to him was Christiaan Huygens.

A major puzzle posed to historians by Mersenne's life and work is this: How did he manage to pursue his free exchange of scientific ideas (in particular, those concerning the ideas of Copernicus) during an era when many others could not (for example, Galileo)? The answer is not clear but may have to do something with Mersenne's style of work and his way of relating to people, and also his close relationship with the Church. (Many centuries after Mersenne, the American essayist and poet Ralph

Waldo Emerson (1803–1882) wrote: "The music that can deepest reach, and cure all ills, is cordial speech". Mersenne would surely have agreed. Probably he could have taught us a thing or two about 'people skills' and 'E.Q.'!)

Mersenne's letters run into thousands of pages.

Mersenne's letters run into thousands of pages. After his death in 1648, letters were found in his cell from seventy eight different correspondents. Read chronologically, they offer a very insightful glimpse at how mathematics and mechanics were evolving during this period of ferment. It is no exaggeration to say that he was the creator of "a scientific academy that stretched across the length and breadth of Europe."

The Cycloid

The cycloid and hyperbola occupied a pride of place in seventeenth century mathematics.

As a case study to illustrate Mersenne's work, we describe his work related to the cycloid. This is the curve traced by a point on a circle which rolls without slipping upon a straight line. It consists of an endless series of arches, called 'cycloidal arches' (see *Figure* 3).

In the first half of the seventeenth century, the cycloid and hyperbola occupied a pride of place in mathematics, for a simple reason: they provided a wonderful laboratory for the testing of new techniques. The circle had been known from ancient times; formulas were known for the area enclosed by a circle and by a segment of a circle. Similarly, results were known (thanks to Archimedes) for the volume and surface area of a sphere; there was little

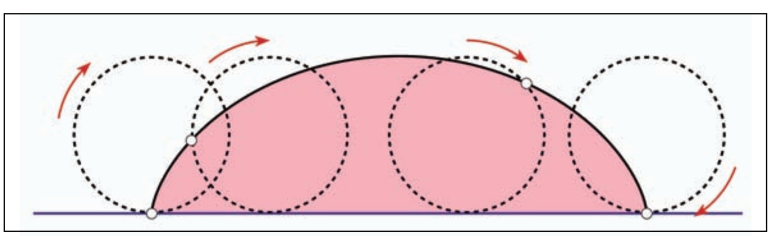


Figure 3. One arch of a cycloid, generated by a circle rolling on a line.

Galileo was one of the first to work on the cycloid.

left to 'test'. Results were also known about the class of objects next higher in complexity: the conic sections. Archimedes had 'squared the parabola': he knew how to find the area of a segment of a parabola. Investigations into the hyperbola had already commenced, and new results were around the corner. So the cycloid and hyperbola were "made to order for the times"; newly discovered techniques could be tested on them: for computing area, using Cavalieri's new theory of 'indivisibles' (which would give way to the 'infinitesimal' of later generations), and for drawing tangents, using the methods due to Fermat and Descartes. Both curves were simple to describe, yet sufficiently complicated that the methods of Euclidean geometry could not unravel all their secrets.

Galileo was one of the first to begin work on the cycloid, and Mersenne probably heard about the curve from him. He wondered whether the cycloid is merely half of an ellipse, 'cut' along its major axis. Galileo determined the area by the simple technique of cutting the cycloidal arch and the generating circle from the same material and comparing their weights (trust the practical minded Galileo to come up with this method, which a 'pure' mathematician would never think of!); he found that the area under the cycloid is roughly three times the area of the circle. That is, if the generating circle has radius r, then the area of the arch is (roughly) $3\pi r^2$.

Mersenne introduced the cycloid to Roberval, who found the area of one arch of the cycloid using the theory of indivisibles.

It was Mersenne who introduced the curve to Roberval and urged him to determine its area. Roberval succeeded in doing so, and showed that Galileo was right: the area is indeed three times that of the generating circle. (Compare this with the estimate for area obtained by supposing the figure to be a semi-ellipse. Since the semi-axes of the ellipse are πr and $2r$, the area is

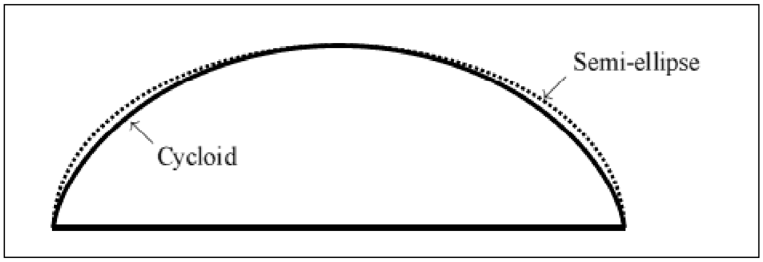


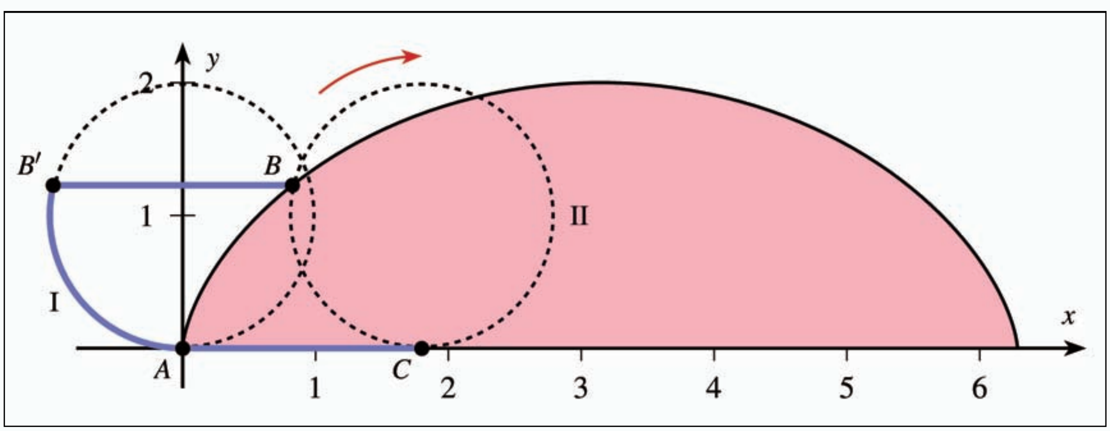
Figure 4. Cycloidal arch and semi-ellipse, superposed.

$\frac{1}{2} \times \pi \times \pi r \times 2r = \pi^2 r^2$. This is about 4.7% larger than Galileo's figure. *Figure* 4 shows the two curves, superposed.)

Quadrature of the Cycloid

The way in which Roberval worked out the area is typical of the times: by using simple, down to earth pre-calculus reasoning, coupled with Cavalieri's method of indivisibles (as distinct from the Greek 'method of exhaustion'). We shall explain his reasoning using modern algebraic language which makes it easier to understand. First, let us determine the equation of a cycloid (see *Figure* 5). Consider two positions (I and II) of the rolling circle in which the point A has moved to B. The movement can be regarded (in the 'physics sense') as the resultant of two movements: a movement along the arc of the circle in position I, from A to B', followed by a

Figure 5. Equation of a cycloid generated by a circle of radius 1 unit.



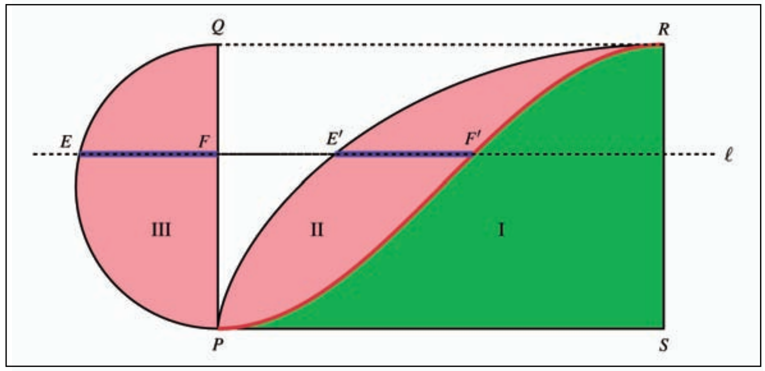


Figure 6. Roberval's construction of the auxiliary curve.

horizontal movement from B' to B. (Note the parallelogram $AB'BC$.) If we denote by t the angle subtended by arc AB' at the centre of circle I, then we easily get the coordinates of B; they are: $(t - \sin t, 1 - \cos t)$. This yields a parametrization of the cycloid (the first arch is generated by $0 \leq t \leq 2\pi$).

Arguing thus, Roberval focuses his attention on half the arch (and half the circle) and constructs an auxiliary curve (see *Figure* 6) essentially by 'subtracting' the sideways movement produced as a result of rolling. To do this he draws a horizontal line through the figure (ℓ), giving rise to the segment EF and the point of intersection E'(on the cycloid). Then he measures off a length $E'F'$ equal to EF. The locus of the F' is the 'auxiliary curve'. Now by invoking Cavalieri's theory of indivisibles he infers that the area of region II enclosed by the auxiliary curve and the cycloid (i.e., by arcs $PF'R$ and $RE'P$) is the same as the area of the semicircle III, i.e., $\frac{1}{2}\pi r^2$. It remains only to find the area of region I.

The curve $PF'R$ is a sinusoidal curve, and it is easy to argue in a number of different ways, by 'physics reasoning' or by invoking the fact that $\cos(\pi/2 - t) = -\cos(\pi/2 + t)$, that the area of region I is half the area of rectangle $PSRQ$; indeed, that the curve $PF'R$

divides the rectangle $PSRQ$ into two congruent halves. Hence the area of region I is $\frac{1}{2}\pi r \times 2r = \pi r^2$. It follows that the area of half the cycloidal arch is $\frac{3}{2}\pi r^2$, and hence that the area of the complete arch is $3\pi r^2$.

Roberval never revealed how he found the area under the cycloid. His method became known only after his death.

Cavalieri's theory of 'indivisibles' is not known by that name now; it is called *Cavalieri's Principle*, and it states that if a family of parallel lines intersects two planar regions in pairs of segments with equal length, then the areas of the two regions are equal. (It has a three dimensional version as well: if a family of parallel planes intersects two solids in pairs of planar regions with equal cross-sectional areas then the volumes of the two solids are equal.)

Roberval did not reveal the method by which he deduced his result. (In fact it became known only after his death.) While this too was typical of the times, in his case it had to do with the fact that he had to justify holding his official position by composing problems for a scholarship competition, and it was therefore in his self-interest to keep his methods secret. Roberval also found a way of finding the slope at any point of a cycloidal arch. (He kept this method a secret too! However in this case it was Descartes who came out with the better solution. But we shall not discuss this problem here.)

It is interesting to report on the tone of the correspondence between Mersenne and Descartes, Fermat and Roberval. Characteristically, Descartes has nothing favourable to say about almost anyone (other than himself, of course). For example he dismisses Roberval's accomplishment with the caustic statement: "Roberval has laboured overmuch to produce so small a result." (Roberval's response: "Prior knowledge of the answer

Descartes cannot have been an easy person to know!

to be found has no doubt been of assistance;" see [6].) Even Fermat was moved to write the following words in a letter dated 1638 to Mersenne: "I will not send you anything else for M. Descartes, since he imposes such harsh regulations on innocent discussions, and it makes me happy to tell you that I have yet to find anyone here who is not of my opinion." Descartes cannot have been a particularly pleasant person to know!

The cycloid continued to feature in front line investigation even much later. For example, the solution to the *brachistochrone problem* (to find the curve of steepest descent; problem posed by Johann Bernoulli in 1696, and solved independently by himself and by Jacob Bernoulli, Newton, Leibnitz and L'Hôpital) turned out to be the cycloid, as did the solution to the *tautochrone problem* (to find the curve for which the time taken by a particle sliding down the curve to its lowest point is independent of its starting point). This is a remarkable economy of nature, and Johann Bernoulli was moved to exclaim, "Nature always tends to act in the simplest way, and so it here lets one curve serve two different functions . . . ."

Mersenne Primes

We close with a brief discussion of the topic for which Mersenne is best remembered in mathematics: Mersenne primes. Interest in such primes dates to Greek times. In Euclid's great text The Elements we find a *perfect number* defined as one for which the sum of the proper divisors equals the number itself. Example: 6; for its proper divisors are $1, 2, 3$, and $1+2+3=6$. Euclid had the following simple and pleasing theorem about such numbers.

Interest in Mersenne primes goes back to Greek times, because of their connection with perfect numbers.

Theorem 1 (Euclid). *Let* $n > 1$ *be a positive integer* *such that* $2^n - 1$ *is a prime number. Then the number* $2^{n-1}(2^n - 1)$ *is perfect.*

Since $2^3 - 1 = 7$ and $2^5 - 1 = 31$ are primes, Theorem 1 leads us to believe that the numbers $2^2 \cdot 7 = 28$ and $2^4 \cdot 31 = 496$ are perfect; and indeed they are.

The proof of Theorem 1 is easy. Let $P = 2^n - 1$; then $P > 1$, and the proper divisors of $2^{n-1}P$ are the numbers $1,\ 2,\ 2^2,\ \ldots,\ 2^{n-1}$ together with $P,\ 2P,\ 2^2P,\ \ldots,\ 2^{n-2}P$. It is easy to check, using the identity $1 + 2 + 2^2 + \cdots + 2^{n-1} = P$, that the sum of the proper divisors is $2^{n-1}P$.

In the middle of the eighteenth century Euler proved that every even perfect number arises from a prime of the form $2^n - 1$. The two results in combination show that the even perfect numbers may be placed in one-to-one correspondence with primes of the form $2^n - 1$. (But this proof is not so simple.)

Mersenne spent a lot of time researching primes of this kind, and in his honour primes of the form $2^n - 1$ are now called *Mersenne primes*. For short let us write $M_n = 2^n - 1$; then M_n is called the n^{th} Mersenne number. It is easy to prove the following: *If* n *is composite, then so is* M_n. Indeed, if a is a proper factor of n then M_a is a proper factor of M_n. (More generally, if x, a, b are positive integers, with $x > 1$ and $b > 1$, then $x^a - 1$ is a proper divisor of $x^{ab} - 1$.) Expressing this in contrapositive form we get the following simple result: *If* M_n *is prime, then* n *is prime.*

Unfortunately the converse of this statement is false; it is not true that M_p is prime for every prime p. Here are the first ten primes p for which M_p is prime:

$$2,\ 3,\ 5,\ 7,\ 13,\ 17,\ 19,\ 31,\ 61,\ 89. \qquad (1)$$

The corresponding Mersenne primes are:

$$M_2 = 3, \quad M_3 = 7, \quad M_5 = 31, \quad M_7 = 127,$$
$$M_{13} = 8191, \quad M_{17} = 131071$$
$$M_{19} = 524287, \quad M_{31} = 2147483647,$$
$$M_{61} = 2305843009213693951,$$
$$M_{89} = 618970019642690137449562111.$$

Note the absence of some small primes in (1): 11, 23, 29 and 37. Here is a beautiful result concerning the factorization of composite Mersenne numbers; it follows from the 'little' theorem of Fermat:

Theorem 2 (Euler). *If p is prime and M_p is composite, then the prime factors of M_p leave remainder* 1 *when divided by* $2p$.

For example, $M_{11} = 2047 = 23 \times 89$; observe that 23 and 89 leave remainder 1 when divided by 22. Similarly, $M_{23} = 8388607 = 47 \times 178481$, and both 47 and 178481 leave remainder 1 when divided by 46. Theorem 2 may have been known to Fermat, but Euler was the first to publish a proof.

(In passing we note that Theorem 2 provides yet another proof of the infinitude of the primes. For any prime number p we consider the number $M_p = 2^p - 1$. Either this number is prime, and it clearly exceeds p; or the number is composite, in which case all its prime divisors exceed $2p$. Both possibilities lead to the existence of a prime number exceeding p. It follows that there is no 'last prime'.)

Theorem 2 provides yet another proof of the infinitude of primes.

One of the tasks that Mersenne set for himself was to find a way to identify the primes p for which M_p is prime.

It is not clear whether he knew Theorem 2 (he probably did know the little theorem of Fermat, as he was in close touch with Fermat), but in 1644 he wrote that $2^p - 1$ is prime for $p = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257$ but composite for the other 44 primes $p < 257$. It is remarkable that he could have made such a statement, for it is surprisingly accurate: there are just three primes which ought to be on his list but are not (61, 89 and 107), and just two which are on his list but ought not to be (67 and 257). It is unclear what the basis of his statement was; there is certainly no direct way he could have checked the primality of M_p for, say, $p > 25$, as the numbers involved are too large. It has been suggested that the few errors in his list may be printer's errors, and that Mersenne may well have been following this rule: *Select only those primes p which differ by* 1 *from a power of* 2*, or by* 3 *from a power of* 4. While this accounts for his list (except for the non-inclusion of 61), it is only a conjecture.

Mersenne primes continue to be a source of investigation; see [7]. It is not known whether or not there are infinitely many such primes. Currently just 48 Mersenne primes are known, the largest being $M_{57885161}$; it was found in February 2013. Recently a conjecture was put forward to which at present no counterexamples have yet been found:

Conjecture. *Let $p > 2$ be a positive integer. If any two of the following statements are true, then so is the third one:*

- *p differs by* 1 *from a power of* 2*, or by* 3 *from a power of* 4*;*
- $2^p - 1$ *is prime;*

- $(2^p + 1)/3$ *is prime.*

Closing Remarks

Mersenne's story is an extraordinary one. It is humbling to realize the strength of community feeling that lies behind his work.

Suggested Reading

[1] http://en.wikipedia.org/wiki/Marin_Mersenne

[2] http://en.wikipedia.org/wiki/File:MarinMersen ne.jpg

[3] http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Mersenne.html

[4] http://gallica.bnf.fr/ark:/12148/bpt6k5471093v

[5] http://www-history.mcs.st-andrews.ac.uk/Biographies/Mersenne.html

[6] J Martin, The Helen of Geometry, *College Mathematics Journal*, Vol.41, No.1, pp.17–27, Jan 2010.

[7] http://en.wikipedia.org/wiki/Mersenne_prime

Address for Correspondence
Shailesh A Shirali
Sahyadri School
Tiwai Hill, Rajgurunagar
Pune 410 513, India.
Email:
shailesh.shirali@gmail.com