The Bakhshāli Square Root Formula

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The Bakhshāli manuscript, which is the oldest surviving document of Indian mathematics, contains a most remarkable formula for estimating square roots. In this article we explore this formula and try to explain why it gives such accurate results. We also search for heuristic ways of deriving the formula, and derive more such formulas.

1. The Bakhshāli Manuscript

The Bakhshāli manuscript (*Figure* 1) is an ancient mathematical work written on birch bark; it is said to be the oldest surviving document of Indian mathematics in South Asia, dating to the 4th century AD. (The dates of the manuscript have been contested over the years, but it is now believed that it belongs to the 4th century or earlier, and that the manuscript found is a copy from the 7th or 8th century of the original one.) It was found in 1881 in the village of Bakhshāli located in north-west Pakistan (about 80 km east of Peshawar), when a stone enclosure within an abandoned building was being dug out. See [1–6] for details. It seems rather a miracle for a readable manuscript to have been found in such a manner. But it is perhaps as much of a miracle that these many centuries later, we not merely comprehend this document but even take delight in it.

What is of great interest is that the Bakhshāli manuscript gives a formula to estimate square roots. Indeed, it is a formula whose accuracy is so great as to be barely credible; the more so, considering the remote time to which it belongs. Elsewhere in the same manuscript (see [1]) there even occur concrete usages of the formula.

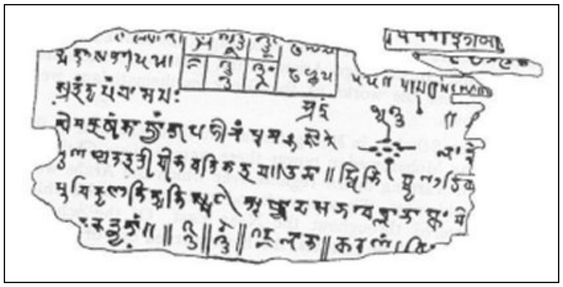

Figure 1. Bakhshāli manuscript.
Source: http://www.daviddarling.info/encyclopedia/B/Bakhshali_Manuscript.html

2. The Bakhshāli Formula

Suppose we wish to find $\sqrt{N}$, where $N > 0$. We express N as $N = A^2 + b$, where $|b|$ is small in comparison with A (as small as possible within the limits of easy computation, using only rational operations, i.e., $+$, $-$, $\times$, $\div$). Using the linear approximation $\sqrt{1+x} \approx 1 + \frac{1}{2}x$ for $x \approx 0$ (this is the tangent approximation) we get:

$$\sqrt{A^2+b} \approx A + \frac{b}{2A}.$$

This formula, which was known to the Babylonians, yields fairly good approximations; e.g., for $\sqrt{10}$ it yields, by taking $A = 3$ and $b = 1$, the fraction $3\frac{1}{6} \approx 3.167$; compare this with $\sqrt{10} \approx 3.162$. But the Bakhshāli formula goes further, by inserting an extra term. It states, in effect:

> *In the case of a non-square number, subtract the nearest square number, divide the remainder by twice this nearest square; half the square of this is divided by the sum of the approximate root and the fraction. This is subtracted and will give the corrected root.*

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After one has deciphered this recipé, it turns to be equivalent to the following formula:

$$\sqrt{A^2+b} \approx A + \frac{b}{2A} - \frac{\left(\frac{b}{2A}\right)^2}{2\left(A + \frac{b}{2A}\right)} \ .$$

For example, if $N = 11$ we may take $A = 3$ and $b = 2$. The formula then gives:

$$3 + \frac{1}{3} - \frac{\frac{1}{9}}{2 \times \left(3 + \frac{1}{3}\right)} = 3 + \frac{1}{3} - \frac{1}{60} = \frac{199}{60} \approx 3.31667.$$

Compare this with the true value: $\sqrt{11} \approx 3.31662$. We see that though b is far from being 'small' in comparison with A, we have still got four decimal place accuracy.

But we can do a lot better than this by taking $A = 3.3$ and $b = 11 - 3.3^2 = 0.11$; note that these numbers are easily obtained using straightforward estimation and rational operations. Since $\frac{b}{2A} = \frac{1}{60}$ we now get the fraction

$$3.3 + \frac{1}{60} - \frac{\frac{1}{3600}}{2 \times \left(3.3 + \frac{1}{60}\right)} = \frac{79201}{23880} \approx 3.316624790,$$

and this agrees with the actual value of $\sqrt{11}$ to *all* the nine decimal places shown!

For $N = 10$, we take $A = 3$, $b = 1$ and get the estimate $721/228 \approx 3.162281$. Compare this with $\sqrt{10} \approx 3.162277$. Better: take $A = 3.2$, $b = 10 - 3.2^2 = -0.24$. Then we get:

$$\sqrt{10} \approx \frac{128009}{40480} \approx 3.162277667,$$

which may be contrasted with the true value of 3.162277660.

The accuracy of the formula is truly astonishing. The obvious question is: *Why does the Bakhshāli formula work so well?* The answer to this is at least accessible to us, as we shall see in the following sections. But there is a much more difficult question: *What path did the authors of the Bakhshāli manuscript take to arrive at their formula?* Sadly, this question will most likely never be answered and remain forever in the domain of surmise.

The accuracy of the Bakhshāli formula is truly astonishing.

3. A Derivation (and Explanation) Using Maclaurin Series

Consider the function $f(x) := \sqrt{1+x}$. Its first few derivatives, evaluated at $x = 0$ (and starting with the zeroth derivative, which is f itself), are:

$$1,\ \frac{1}{2},\ -\frac{1}{4},\ \frac{3}{8},\ -\frac{15}{16},\ \frac{105}{32},\ \ldots .$$

We seek an easily computable function which agrees with $f(x)$ in its derivatives at $x = 0$; as many derivatives as possible. Why would we want this? Here is a heuristic explanation. If two smooth curves intersect at a point P and share the same slope at that point, then in the vicinity of P they will stay close together. This follows from the very definition of equality of slope. But if their second derivatives differ at P, then this will lead to a gradually widening gap in their first derivatives as we move away from P, and in turn to a widening gap between the curves themselves. Recursively carrying this logic forward, we expect that two curves which agree in their first several derivatives at some point (starting with the zeroth derivative) will stay close together over a larger domain surrounding that point.

We seek an easily computable function which agrees with $f(x)$ in its derivatives at $x = 0$.

The most easily computed functions are the *polynomials with rational coefficients*, and the polynomials of successively higher degrees which agree with $f(x)$ in its successive derivatives at $x = 0$ (which, fortunately for us, are all rational numbers) are the partial sums of the Maclaurin series of $f(x)$ about $x = 0$:

$$1, \quad 1+\frac{x}{2}, \quad 1+\frac{x}{2}-\frac{x^2}{8}, \quad 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}, \quad \ldots .$$

The graphs of these functions, for $-0.5 \le x \le 2.5$ are shown in *Figure* 2, together with the graph of $\sqrt{1+x}$. Their closeness in the vicinity of $x = 0$ is very visible, as also the growing error as x goes beyond 1.

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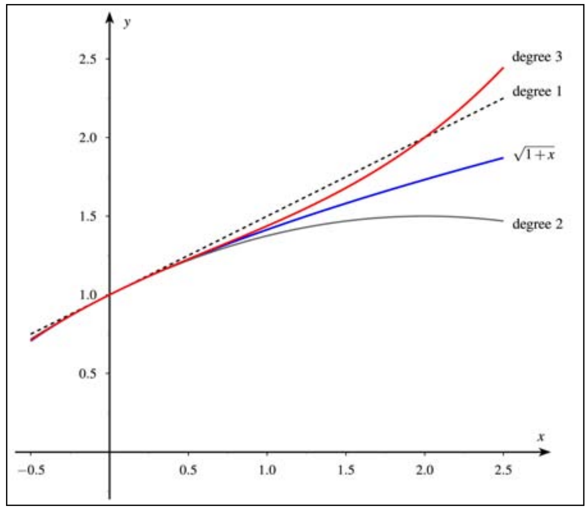

Figure 2. Graphs of $\sqrt{1+x}$ and the first few partial sums of its Maclaurin series about $x = 0$.

Next in line are the rational functions with rational coefficients.

Next in line are the *rational functions with rational coefficients*, which are just as easily computed; they have the form (p/q), where p and q are polynomials (with rational coefficients). Curiously, these turn out to be more efficient than polynomials, in terms of the accuracy achieved while keeping the degree as small as possible. We demonstrate this by finding rational functions $r(x)$ of the following types which agree with $f(x)$ in their first several derivatives at $x = 0$ (note that there is automatic agreement in the zeroth derivative, by the choice in the form of the function):

$$\frac{1+ax}{1+bx}, \quad \frac{1+ax}{1+bx+cx^2}, \quad \frac{1+ax+bx^2}{1+cx}.$$

Their successive derivatives at $x = 0$ are shown in *Table* 1.

Table 1.

$r(x)$	$r^{(1)}(0)$	$r^{(2)}(0)$	$r^{(3)}(0)$
$\frac{1+ax}{1+bx}$	$a-b$	$-2(a-b)b$	(not needed)
$\frac{1+ax}{1+bx+cx^2}$	$a-b$	$-2(ab-b^2+c)$	$-6(ab^2-b^3-ac+2bc)$
$\frac{1+ax+bx^2}{1+cx}$	$a-c$	$2(b-ac+c^2)$	$-6c(b-ac+c^2)$

Equating the expressions in the table with the appropriate derivatives of $f(x)$ at $x = 0$ and solving for the unknown coefficients, we get the following rational functions:

$$r_{1,1}(x) := \frac{1+\frac{3}{4}x}{1+\frac{1}{4}x}, \qquad r_{1,2}(x) := \frac{1+\frac{5}{6}x}{1+\frac{1}{3}x-\frac{1}{24}x^2},$$

$$r_{2,1}(x) := \frac{1+x+\frac{1}{8}x^2}{1+\frac{1}{2}x}.$$

It is of interest to make a comparison of the graphs of these three functions with that of $\sqrt{1+x}$, over some suitable interval, say $-0.5 \le x \le 2.5$; see *Figure* 3. We see right away that these curves stay very much closer to the curve $\sqrt{1+x}$ than do the corresponding curves obtained from the Maclaurin series (for which a fair degree of divergence is visible even for $x = 1$).

From the graph we also see that of the above three functions, the one that most closely approximates $\sqrt{1+x}$ is $r_{2,1}(x)$. But this is *just* the Bakhshāli approximation with $A = 1$ and $b = x$, as may be seen from the following easily checked algebraic identity:

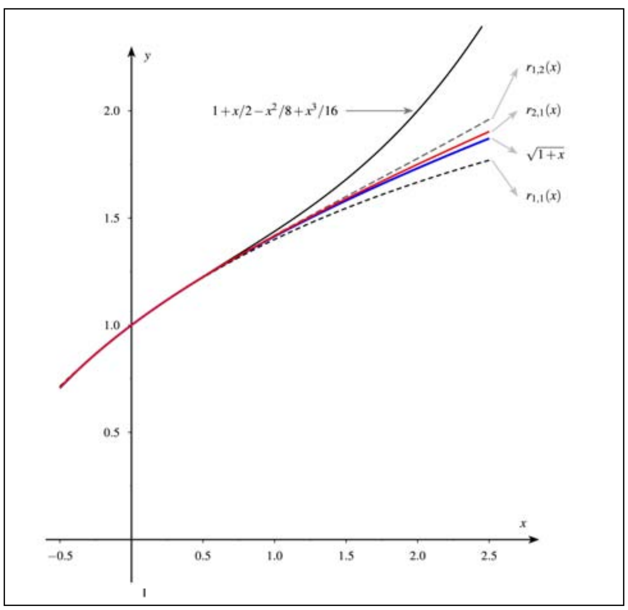

Figure 3. Graphs of $\sqrt{1+x}$ and the functions $r_{1,1}(x)$, $r_{1,2}(x)$ and $r_{2,1}(x)$.

$$1+\frac{x}{2}-\frac{\left(\frac{x}{2}\right)^2}{2\left(1+\frac{x}{2}\right)} = \frac{1+x+\frac{x^2}{8}}{1+\frac{x}{2}} \ .$$

In Passing, a Closer Look

We commented that $r_{2,1}(x)$ is better than $r_{1,2}(x)$ as an approximation for $\sqrt{1+x}$. Can we explain this in some way? By design, the three functions agree in their zeroth, first, second and third derivatives at $x=0$. Let us therefore compare their *fourth* derivatives at $x=0$. We find the following:

$$f^{(4)}(0) = -\frac{15}{16}, \qquad r_{1,2}^{(4)}(0) = -\frac{5}{8}, \qquad r_{2,1}^{(4)}(0) = -\frac{3}{4}.$$

Hence we have the following relations for $x \approx 0$, which also tell us the magnitudes of the errors involved:

$$r_{1,2}(x) - f(x) \approx \frac{5x^4}{384}, \qquad r_{2,1}(x) - f(x) \approx \frac{x^4}{128}, \qquad \frac{r_{2,1}(x) - f(x)}{r_{1,2}(x) - f(x)} \approx \frac{3}{5}.$$

Thus for $x \approx 0$, the error in $r_{2,1}(x)$ is approximately $(3/5)$ of the error in $r_{1,2}(x)$, and in the same direction.

4. A Derivation Using the Newton–Raphson Iteration

This derivation too is calculus-based, and it is surely the most efficient way of deriving the Bakhshāli formula; it illustrates in a striking way the power of the Newton–Raphson method for the numerical solution of equations. The idea is: *If $g(x)$ is a smooth function defined on $\mathbb{R}$, and x_0 is a close approximation to the solution of the equation $g(x)=0$, then a closer approximation is the quantity*

$$x_1 = x_0 - \frac{g(x_0)}{g'(x_0)} \ .$$

It illustrates in a striking way the power of the Newton–Raphson method for the numerical solution of equations.

As we need $\sqrt{A^2+b}$, we take $g(x) = x^2 - (A^2+b)$. Take x_0 to be:

$$x_0 = A + \frac{b}{2A}.$$

This gives: $g(x_0) = \frac{b^2}{4A^2} = \left(\frac{b}{2A}\right)^2$, $g'(x_0) = 2\left(A + \frac{b}{2A}\right)$, and:

$$x_1 = A + \frac{b}{2A} - \frac{\left(\frac{b}{2A}\right)^2}{2\left(A+\frac{b}{2A}\right)}.$$

This is exactly the Bakhshāli formula. The ease with which we have arrived at the formula is impressive but we have lost out on the many insights we got in the previous section. Some gain, some loss ....

5. Another Derivation Using Iteration

As noted in [4], one can arrive at the Bakhshāli formula from the tangent approximation,

$$\sqrt{A^2+b} \approx A + \frac{b}{2A},$$

simply by iteration – with no recourse to derivatives, Maclaurin series and the like.

To see how, let N be any positive number whose square root is needed. Let A be an approximation to the square root, and let b be a measure of the error in A:

$$b = N - A^2.$$

We replace the approximation A by its first order approximation A',

$$A' = A + \frac{b}{2A},$$

and compute the new error term b' by $b' = N - (A')^2$. We thus have a map:

$$(A, b) \longmapsto (A', b').$$

We may describe the map explicitly in terms of the arguments A and b: since

$$b' = N - (A')^2 = A^2 + b - \left(A + \frac{b}{2A}\right)^2 = -\frac{b^2}{4A^2},$$

Could this be the path which was taken by the original discoverers of the formula? Possibly, but we will never know.

the map is:

$$(A, b) \longmapsto \left(A + \frac{b}{2A},\ -\frac{b^2}{4A^2}\right).$$

Applying the map a second time we get:

$$\begin{aligned}(A, b) &\longmapsto \left(A + \tfrac{b}{2A},\ -\tfrac{b^2}{4A^2}\right)\\ &\longmapsto \left(A + \tfrac{b}{2A} - \tfrac{b^2}{4A^2}\left(\tfrac{1}{2\left(A+\frac{b}{2A}\right)}\right),\ \ldots\right),\end{aligned}$$

where we have not bothered to simplify the second component ('...') of the second iterate, as it is not needed. The first component of this iterate simplifies to

$$A + \frac{b}{2A} - \frac{b^2}{4A^2}\left(\frac{1}{2\left(A + \frac{b}{2A}\right)}\right) = A + \frac{b}{2A} - \frac{\left(\frac{b}{2A}\right)^2}{2\left(A + \frac{b}{2A}\right)},$$

and we see that we have hit upon the Bakhshāli formula. Could *this* be the path which was taken by the original discoverers of the formula? Possibly, but we will never know ....

Remark. Reference [4] gives yet another natural way in which the Bakhshāli formula may be derived: from the continued fraction expansion, $\sqrt{A^2 + b} = A + \frac{b}{2A+}\,\frac{b}{2A+}\,\frac{b}{2A+}\cdots$ (it is the third convergent).

6. Improvement on the Bakhshāli Formula

We have seen that by iterating the relation $\sqrt{1+x} \approx 1 + (1/2)x$ using an appropriate starting point, we hit upon the Bakhshāli formula. We now show that a small change allows us to do still better. Recall that the fractional linear form $(1 + \frac{3}{4}x)/(1 + \frac{1}{4}x)$ is a better approximation to $\sqrt{1+x}$ than is $1 + (1/2)x$. Why not, then, use *this* approximation as a starting point for the iteration?

The 'b'-term for this approximation (using the notation of Section 5) is

$$1 + x - \left(\frac{1 + \frac{3}{4}x}{1 + \frac{1}{4}x}\right)^2 = \frac{x^3}{(4+x)^2}.$$

Hence the map $A \longmapsto A + \frac{b}{2A}$ yields as the next approximation:

$$\frac{1+\frac{3}{4}x}{1+\frac{1}{4}x} + \frac{x^3}{(4+x)^2}\frac{1}{2}\left(\frac{1+\frac{3}{4}x}{1+\frac{1}{4}x}\right)^{-1} = \frac{(x+2)\left(x^2+16x+16\right)}{2(x+4)(3x+4)} = \frac{\left(1+\frac{x}{2}\right)\left(1+x+\left(\frac{x}{4}\right)^2\right)}{\left(1+\frac{1}{4}x\right)\left(1+\frac{3}{4}x\right)}.$$

This yields the following Bakhshāli-type result:

$$\sqrt{A^2+b} \approx A + \frac{b}{2A} - A\cdot\left(\frac{b}{A^2}\right)^2 \cdot \frac{\left(1+\frac{1}{2}\frac{b}{A^2}\right)}{\left(1+\frac{b}{A^2}+\frac{3}{16}\frac{b^2}{A^4}\right)}\ .$$

To illustrate the accuracy of this formula we estimate $\sqrt{5}$; if we take $A=2$, $b=1$ we get the fraction $2889/1292$, which is in error by roughly 10^{-7}; and if we take $A=2.2$, $b=0.16$ we get the fraction $930249/416020$, which differs from $\sqrt{5}$ by roughly 10^{-12}.

Computation of the Maclaurin series about $x=0$ allows us to account for the extremely high degree of accuracy of the modified Bakhshāli formula:

$$\frac{\left(1+\frac{x}{2}\right)\left(1+x+\left(\frac{x}{4}\right)^2\right)}{\left(1+\frac{1}{4}x\right)\left(1+\frac{3}{4}x\right)} - \sqrt{1+x} = \frac{x^6}{2048} + \cdots .$$

The leading term of the corresponding series for the original Bakhshāli formula (i.e., for $r_{2,1} - \sqrt{1+x}$) is $x^4/128$, which has degree 4, and here it has degree 6; so the accuracy level has gone up by two orders of magnitude.

By playing around with these ideas we may obtain more functions of this kind, but they come at a price: the formulas start to get unwieldy. Here is one such:

$$\sqrt{A^2+b} \approx A + \frac{b}{2A} - \frac{\left(\frac{b}{2A}\right)^2}{2\left(A+\frac{b}{2A}\right)} - \frac{b^4}{8A\left(2A^2+b\right)\left(8A^4+8bA^2+b^2\right)}.$$

Suggested Reading

[1] Swami S P Sarasvati and Dr Usha Jyotishmati, *The Bakhshāli Manuscript*, Dr. Ratna Kumari Svadhyaya Sansthan, Allahabad. This book is difficult to get in print but may be read online at the following: http://www.scribd.com/doc/75830711/The-Bakhshali-Manuscript

[2] The Bakhshāli Manuscript. http://www-history.mcs.st-and.ac.uk/HistTopics/Bakhshāli_ manuscript.html

[3] The Bakhshāli Manuscript. http://www-history.mcs.st-and.ac.uk/Projects/Pearce/Chapters/Ch6.html

[4] M N Channabasappa, *Square Root Formula in Bakhshāli Manuscript*. http://www.math10.com/en/maths-history/math-history-in-india/Bakhshali/bakshali.html

[5] C N Srinivasiengar, *The History of Ancient Indian Mathematics*, Calcutta, World Press, 1967.

[6] http://en.wikipedia.org/wiki/Bakhshali_manuscript

[7] http://www.daviddarling.info/encyclopedia/B/Bakhshali_ Manuscript.html

7. A Bakhshāli-Type Formula for Cube Roots

We may readily extend the above idea to higher order roots. For example, for cube roots we use the basic relation:

$$(1+x)^{1/3} \approx 1 + \frac{x}{3} \qquad (x \approx 0)$$

to deduce that

$$\left(A^3+b\right)^{1/3} \approx A + \frac{b}{3A^2} \qquad (|b| \ll A).$$

This yields the following mapping for cube roots which we may use iteratively, just as we did in Section 5:

$$(A,b) \longmapsto \left(A + \frac{b}{3A^2},\ -\frac{b^2\left(9A^3+b\right)}{27A^6}\right),$$

where the second coordinate on the right is $A^3+b-\left(A+b/(3A^2)\right)^3$. Iterating this map we get the following relation:

$$\left(A^3+b\right)^{1/3} \approx A + \frac{b}{3A^2} - \left(\frac{b}{3A^2}\right)^2 \frac{3A+\frac{b}{3A^2}}{3\left(A+\frac{b}{3A^2}\right)^2}.$$

How good is the formula? Let us check the outcome for $9^{1/3}$, with $A=2$, $b=1$; we get:

$$2+\frac{1}{12}-\frac{1}{12^2}\,\frac{6+\frac{1}{12}}{3\left(2+\frac{1}{12}\right)^2} = \frac{23401}{11250} \approx 2.080088.$$

In fact, $9^{1/3} \approx 2.080083823$. We have not done too badly; and we can easily do much better (e.g., try $A=2.1$, $b=9-2.1^3=-0.261$; we get 2.080083840).

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