A Cube as (Almost) a Sum of Two Cubes

A Remarkable Identity of S Ramanujan

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It is known that one cannot write an integer cube as a sum of two integer cubes (Fermat's Last Theorem). The number 1728 ($=12^3$) comes close to being the sum of two cubes, but falls short by 1. An entry in Srinivasa Ramanujan's *Lost Notebook* gives a remarkable identity which provides infinitely many such examples. This article discusses a proof of this identity, as also another similar identity.

In a very famous margin note Pierre de Fermat wrote "*... it is not possible to split a cube as a sum of two cubes* ...." Well, if one cannot split a cube as a sum of two cubes, how close can one get? Recall the famous *Srinivasa Ramanujan, G H Hardy and the Taxi cab number* incident.

I will not retell the story but am only interested in the *number* and its property:

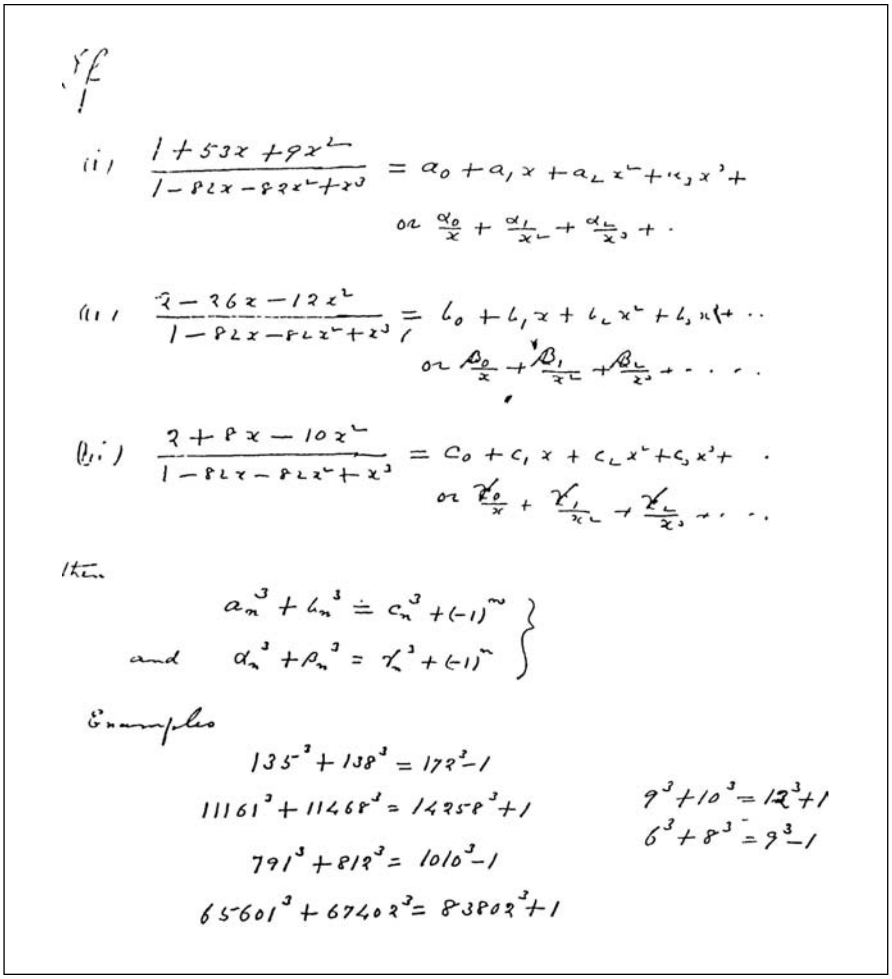
$$1729 = 9^3 + 10^3 = 12^3 + 1.$$

In the above example $12^3 = 1728$ is 'almost' a sum of two cubes; it falls short by 1: $1728 = 9^3 + 10^3 - 1$. Are there other examples? Ramanujan seems to have pondered over this question and on p. 341 of the *Lost Notebook* [1], one finds, in his own handwriting, the following entry (*Figure* 1).

1. An Earlier, Related Identity

To give a flavour of the kind of algebra involved we shall present a similar identity which appeared as Problem no. 2078 in *Nouvelles Annales de Mathématiques* in 1907, proposed by R Amsler and for which a solution was presented in the same journal in 1916 by M L Chanzy [2].

Keywords
Recurrence relations, (power series) expansions, polynomials, quadratic forms.



If

(i) $$\frac{1+53x+9x^2}{1-82x-82x^2+x^3} = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$
$$\text{or } \frac{\alpha_0}{x} + \frac{\alpha_1}{x^2} + \frac{\alpha_2}{x^3} + \cdots$$

(ii) $$\frac{2-26x-12x^2}{1-82x-82x^2+x^3} = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots$$
$$\text{or } \frac{\beta_0}{x} + \frac{\beta_1}{x^2} + \frac{\beta_2}{x^3} + \cdots$$

(iii) $$\frac{2+8x-10x^2}{1-82x-82x^2+x^3} = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$$
$$\text{or } \frac{\gamma_0}{x} + \frac{\gamma_1}{x^2} + \frac{\gamma_2}{x^3} + \cdots$$

then

$$\left.\begin{aligned} a_n^3 + b_n^3 &= c_n^3 + (-1)^n \\ \text{and} \quad \alpha_n^3 + \beta_n^3 &= \gamma_n^3 + (-1)^n \end{aligned}\right\}$$

Examples

$$135^3 + 138^3 = 172^3 - 1$$
$$11161^3 + 11468^3 = 14258^3 + 1$$
$$791^3 + 812^3 = 1010^3 - 1$$
$$65601^3 + 67402^3 = 83802^3 + 1$$

$$9^3 + 10^3 = 12^3 + 1$$
$$6^3 + 8^3 = 9^3 - 1$$

Figure 1.

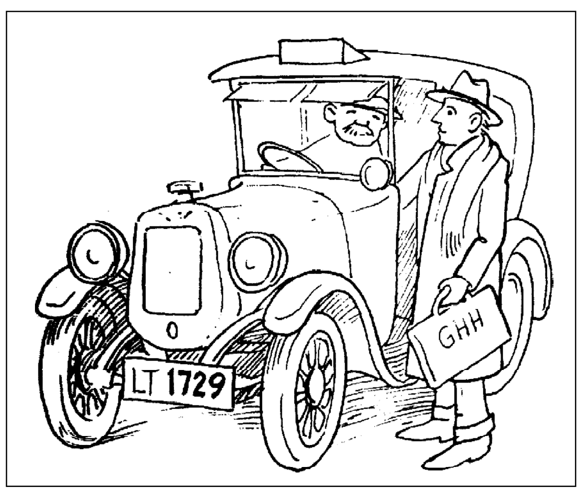


Figure 2. This was created by E A Lord at my request in 1998. While on a vacation in London, he consulted magazines of the 1910's to get an idea of how cabs and cabbies looked like then!

We shall reproduce his solution here. (Translation of both problem and solution from original French is mine; notations have been retained from the original.)

Problem No. 2078 (R Amsler, 1907). Let u_n denote the nth coefficient in the expansion of

$$\frac{1}{1-3x-3x^2-x^3}$$

and v_n, the nth coefficient in the expansion of

$$\frac{1}{1+3x+3x^2-x^3}.$$

If $X = u_{n+1}$, $Y = u_n + u_{n+1}$, $Z = v_n$ and $T = v_n + v_{n+1}$, then show that

$$2(X^3 + Z^3) = Y^3 + T^3.$$

Solution (M L Chanzy, 1916). Let $f(x) = hx^3 + px^2 + qx + r$ be a cubic polynomial with three distinct roots and let

$$-\frac{1}{f(x)} = \sum_{n\geq 0} u_n x^n, \qquad \frac{1}{x^3 f(1/x)} = \sum_{n\geq 0} v_n x^n.$$

Then we have the following identity:

$$\left(\frac{r}{h}\right)^{2n+4} u_{n+1}^3 f\left(\frac{u_n}{u_{n+1}}\right) + v_n^3 f\left(\frac{v_{n+1}}{v_n}\right) = 0. \qquad (1)$$

Assuming this identity for the moment, apply it to the polynomial $f(x) = (x+1)^3 - 2$ to get the desired result:

$$(u_n + u_{n+1})^3 - 2u_{n+1}^3 + (v_n + v_{n+1})^3 - 2v_n^3 = 0.$$

Now for the proof of the identity itself. Let a, b, c be the three roots of $f(x)$. We have

$$-\frac{1}{f(x)} = \frac{-1}{h(x-a)(x-b)(x-c)}$$

$$= \frac{1}{h(a-b)(b-c)(c-a)}\left(\frac{b-c}{x-a}+\frac{c-a}{x-b}+\frac{a-b}{x-c}\right)$$

$$= \frac{1}{h(a-b)(b-c)(c-a)}\left[\frac{c-b}{a}\left(\sum_{n\geq 0}\frac{x^n}{a^n}\right)\right.$$

$$\left.+\frac{a-c}{b}\left(\sum_{n\geq 0}\frac{x^n}{b^n}\right)+\frac{b-a}{c}\left(\sum_{n\geq 0}\frac{x^n}{c^n}\right)\right]$$

Thus we get

$$u_n = \frac{1}{h(a-b)(b-c)(c-a)}\left[\frac{c-b}{a^{n+1}}+\frac{a-c}{b^{n+1}}+\frac{b-a}{c^{n+1}}\right]. \tag{2}$$

Similarly,

$$\frac{1}{x^3 f(1/x)} = \frac{1}{h(1-ax)(1-bx)(1-cx)}$$

$$= \frac{-1}{h(a-b)(b-c)(c-a)}$$

$$\times\left(\frac{(b-c)a^2}{1-ax}+\frac{(c-a)b^2}{1-bx}+\frac{(a-b)c^2}{1-cx}\right)$$

$$= \frac{-1}{h(a-b)(b-c)(c-a)}$$

$$\times\left[(c-b)a^2\left(\sum_{n\geq 0}a^n x^n\right)+(a-c)b^2\right.$$

$$\left.\times\left(\sum_{n\geq 0}b^n x^n\right)+(b-a)c^2\left(\sum_{n\geq 0}c^n x^n\right)\right].$$

Therefore

$$v_n = \frac{-1}{h(a-b)(b-c)(c-a)}$$

$$\times\left[(b-c)a^{n+2}+(c-a)b^{n+2}+(a-b)c^{n+2}\right].$$

We have

$$
\begin{aligned}
u_n - au_{n+1} &= \frac{1}{h(a-b)(b-c)(c-a)} \\
&\quad \times \left[\frac{a-c}{b^{n+1}} + \frac{b-a}{c^{n+1}} - a\frac{a-c}{b^{n+1}} - a\frac{b-a}{c^{n+1}}\right] \\
&= \frac{1}{h(b-c)}\frac{c^{n+2}-b^{n+2}}{(bc)^{n+2}}
\end{aligned}
$$

and also,

$$v_{n+1} - av_n = \frac{-1}{h(b-c)}(c^{n+2} - b^{n+2}).$$

Thus,

$$u_n - au_{n+1} = -\frac{(v_{n+1} - av_n)}{(bc)^{n+2}}.$$

Similarly,

$$
\begin{aligned}
u_n - bu_{n+1} &= -\frac{(v_{n+1} - bv_n)}{(ac)^{n+2}}, \\
u_n - cu_{n+1} &= -\frac{(v_{n+1} - cv_n)}{(ab)^{n+2}}.
\end{aligned}
$$

Multiplying the last three equations, we get

$$u_{n+1}^3 f\left(\frac{u_n}{u_{n+1}}\right) = -\frac{v_{n+1}^3 f\left(\frac{v_{n+1}}{v_n}\right)}{(abc)^{2n+4}}.$$

Noting that $abc = -r/h$ completes the proof of the identity (1).

Chanzy also notes that by taking $f(x) = (x+1)^3 - k$, we have:

$$(k-1)^{2n+4}[(u_n+u_{n+1})^3 - ku_{n+1}^3] + (v_n+v_{n+1})^3 - kv_n^3 = 0.$$

This, for $n = 3m+1$, $m \geq 0$, gives solutions of

$$X^3 + Z^3 = k(y^3 + T^3)$$

thus generalising Amsler's problem.

2. Ramanujan's Identity

Ramanujan's identity, involves recurrence relations in addition to the above algebra. The above example was given to show that such problems were being considered then. Is it possible that he had seen/heard of this problem of Amsler's? Let us recall the identity to be proved: If

$$\frac{1+53x+9x^2}{1-82x-82x^2+x^3} = \sum_{n\geq 0} a_n x^n \ ,$$

$$\frac{2-26x-12x^2}{1-82x-82x^2+x^3} = \sum_{n\geq 0} b_n x^n \ ,$$

and

$$\frac{2+8x-10x^2}{1-82x-82x^2+x^3} = \sum_{n\geq 0} c_n x^n \ ,$$

then

$$a_n^3 + b_n^3 = c_n^3 + (-1)^n.$$

Proof. This proof follows the one given by M D Hirschhorn [3]. Put $f(x) = x^3 - 82x^2 - 82x + 1$. Let us use the above method to get the expansion of $\frac{1}{f(x)}$. First, note that $x = -1$ is a root of $f(x)$ and we have:

$$f(x) = (x+1)(x^2 - 83x + 1) \ .$$

Thus, if α, β are the roots of the quadratic factor, we have $\alpha + \beta = 83$ and $\alpha\beta = 1$. Therefore, if u_n denotes the nth coefficient of $\frac{1}{f(x)}$, then from (2), taking $a = -1, b = \alpha, c = \beta$, we have, after simplification, (note that absence of the 'minus' sign and therefore the resulting interchange of the terms in the denominator of the first factor in the expression for u_n):

$$\begin{aligned}(b-a)(c-b)(a-c) &= (\alpha+1)(\beta-\alpha)(-\beta-1) \\ &= (\alpha-\beta)(\alpha\beta+\alpha+\beta+1) \\ &= 85(\alpha-\beta)\end{aligned}$$

and

$$u_n = \frac{1}{85}\left((-1)^n + \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} + \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta}\right).$$

Further, since α, β are the roots of $(x^2 - 83x + 1)$, we have $\alpha^2 = 83\alpha - 1$ and $\beta^2 = 83\beta - 1$. Thus, u_n will be, on simplification, a linear combination of $-1, \alpha, \beta$. It is fun to compute a few values of u_n: $u_0 = 1$, $u_1 = 82$, $u_2 = 6806$, $u_3 = 564815$.

The expressions for a_n, b_n and c_n are, for $n \geq 2$ (a_0, a_1 can be obtained directly):

$$\begin{aligned} a_n &= u_n + 53u_{n-1} + 9u_{n-2}, \\ b_n &= 2u_n - 26u_{n-1} - 12u_{n-2}, \\ c_n &= 2u_n + 8u_{n-1} - 10u_{n-2}. \end{aligned}$$

Using the values of u_n given above we have

$$a_0 = 1, b_0 = 2, c_0 = 2; \quad a_1 = 135, b_1 = 138, c_1 = 172;$$

$$a_2 = 11161, b_2 = 11468, c_2 = 14258.$$

What we said about u_n's, that they are linear combinations of $-1, \alpha, \beta$, holds true for a_n, b_n and c_n as well. This, in turn will imply that a_n^3, b_n^3 and c_n^3 are linear combinations of the following seven quantities:

$$\begin{aligned} (-1)^3 = -1, \alpha^3, \beta^3, \alpha^2\beta = \alpha(\alpha\beta) &= \alpha, \alpha\beta^2 = \beta(\alpha\beta) \\ &= \beta, (-1)\alpha^2, (-1)\beta^2. \end{aligned}$$

Let us consider the series:

$$\sum_{n \geq 0} \left(a_n^3 + b_n^3 - c_n^3 - (-1)^n\right) x^n \ .$$

It follows that this series is of the form $N(x)/D(x)$ where $D(x)$ is the following polynomial of degree 7:

$$(1+x)(1-\alpha^3 x)(1-\beta^3 x)(1-\alpha x)(1-\beta x)(1+\alpha^2 x)(1+\beta^2 x)$$

and $N(x)$ is a polynomial of degree at most 6. Suppose that $N(x) = d_0 + d_1x + d_2x^2 + d_3x^3 + d_4x^4 + d_5x^5 + d_6x^6$. Thus, if we put $a_n^3 + b_n^3 - c_n^3 - (-1)^n = q^n$, we have

$$\sum_{n\geq 0} q_n x^n = \frac{d_0 + d_1x + d_2x^2 + d_3x^3 + d_4x^4 + d_5x^5 + d_6x^6}{D(x)}.$$

Observe now that if $q_0 = 0$, then $d_0 = 0$; if $q_0 = 0, q_1 = 0$, then $d_0 = 0, d_1 = 0$, and so on. Therefore, if $q_n = 0$ for $0 \leq n \leq 6$, then $N(x)$ is identically 0 and so, it will follow that $q_n = 0$ for all n. Hirschhorn ([3], p.3) has verified that this is indeed so. It is a good exercise to carry out this verification computing u_n for $0 \leq n \leq 6$.

3. Conclusion

One wonders how Ramanujan was able to write down such identities. Some work had been done by Hirschhorn [4] and Craig [5] on relating this identity to some identities by Ramanujan involving quadratic forms. Hirschhorn [6] has also found the following interesting matrix relation of this identity:

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = \begin{pmatrix} 63 & 104 & -68 \\ 64 & 104 & -67 \\ 80 & 131 & -85 \end{pmatrix}^n \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}.$$

Suggested Reading

[1] S Ramanujan, *The Lost Notebook and Other Unpublished Papers*, p.341, Narosa, New Delhi, 1988.

[2] M L Chanzy, Solution (to a problem of R Amsler), *Nouv. Ann. Math.*, Vol.16, 282285, 1916.

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[5] M Craig, Cubic Surfaces and Linear Recurrences, *Austral. Math. Soc. Gazette*, Vol.32 No.4, September 2005.

[6] M D Hirschhorn, Ramanujan and Fermat's Last Theorem, *Austral. Math. Soc. Gazette.*, Vol.31, pp.256–257, 2004.

[7] G H Hardy and E M Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, 1980.

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