Classroom

In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

Analysing Spherical Aberration in Concave Mirrors

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What is the real behaviour of the spherically concave mirror which, we were taught in school, focuses a parallel beam to a point? What is the accuracy and range of its focussing behaviour and what happens outside that range? How do we relate this behaviour to the measurable dimensions of the mirror? This article presents answers to these questions using simple geometry and algebra.

1. Introduction and Motivation

We learn in high school physics that concave mirrors focus parallel beams to their foci and that rays emanating from the foci are reflected from the mirror as a parallel beam. These concave mirrors are drawn, as shown in *Figure* 1, as arcs of circles of generous and uniform curvature [1,2].

We later learn in college that automobile headlamp reflectors are parabolic [3]. College geometry teaches us that of the conic sections, the parabola is one which has a point called the focus from which if light rays are made to emanate and hit the parabolic surface, they

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Geometrical optics, spherical aberration.

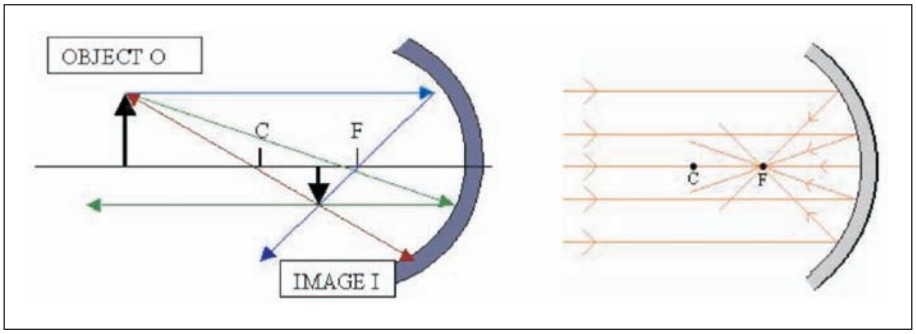


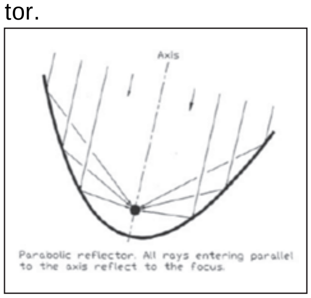
Figure 1. Classic depiction of concave mirror in physics textbooks, with an arc that subtends a large angle upon the centre.

would bounce off parallel to one another (*Figure* 2).

How can two surfaces with different equations (and therefore different geometries) possess the same property when it comes to the reflection of light from their surfaces? This question arose when a student of the author, studying solar water purifiers, brought up the issue of the various shapes of mirrors he had studied. These included 'parabolic reflectors' and 'spherical bowl reflectors.'

In this article we show that the behaviour of spherically concave mirrors resembles that of parabolic mirrors only within a range of sizes parametrised by the ratio of rim diameter and depth of the concave mirror. It is within this range that the incident (or emergent) ray lies close enough to the optical axis of the system to give us the 'paraxial approximation' which causes the observed reflection in spherical mirrors to tally close enough with that of parabolic mirrors. Outside this range, the paraxial approximation does not hold and one gets the well-known phenomenon of spherical aberration in spherical mirrors which parabolic mirrors are free of.

Figure 2. Parabolic reflector.



2. Concave Mirror Geometry

A spherically concave mirror possesses two points of special interest – the focal point F and the centre of curvature C. The focal point lies, as shown in *Figure* 1, at half the distance of the centre of curvature from the mirror, and has the special property that light rays passing

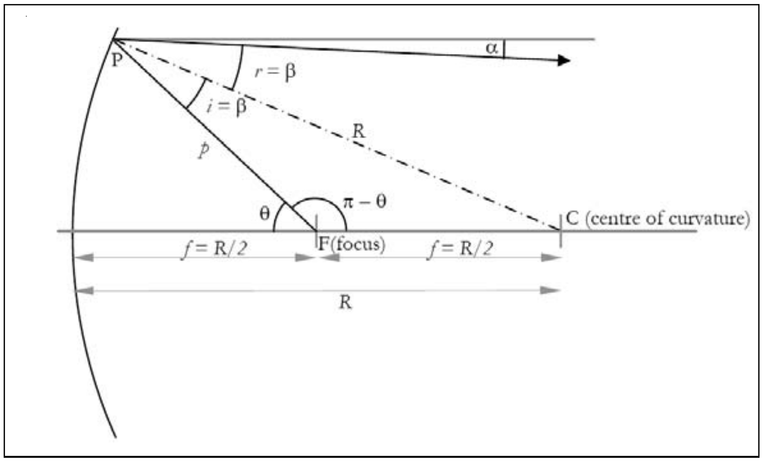


Figure 3. Geometry of ray reflection through a point away from the principal axis.

through it emerge parallel to each other upon reflection. Light rays passing through the centre of curvature bounce back on themselves.

The question studied in this article is whether *any* ray passing through F as shown in *Figure* 3 will actually be reflected parallel to the principal axis as claimed in school physics textbooks. A simple geometric analysis, where the normal to the mirror's reflecting surface at the point of incidence of the light beam is shown as PC, will help to answer that problem.

The diagram in *Figure* 3 is drawn to scale; so it is visually evident that following the laws of reflection, the reflected ray is not parallel to the principal axis. This deviation from parallelism is called 'spherical aberration', a well-documented phenomenon [4].

We now study the behaviour of a light beam incident upon the mirror at a smaller distance from the principal axis (*Figure* 4). In this case, the reflected ray is quite close to being parallel to the principal axis. Let us study this change quantitatively by observing how far the incident beam can be from the principal axis before its parallelism with the axis starts to become questionable. This will lead to answering the question of how large a spherical mirror can be before it ceases to be a

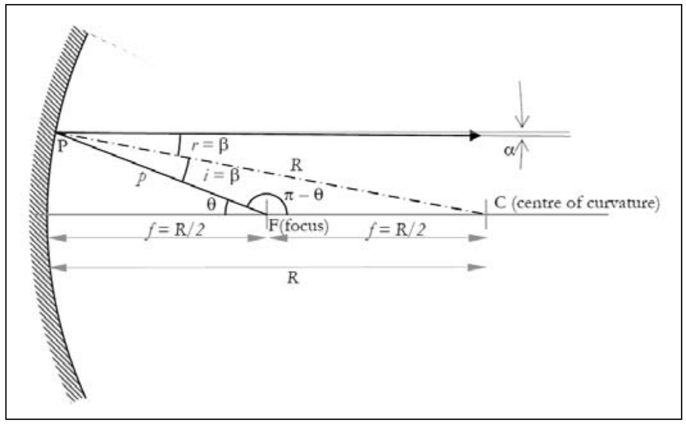


Figure 4. Geometry of ray reflection through a point near the principal axis.

parallel-beam reflector, i.e., at what size of the mirror does spherical aberration set in.

Let the angle of deviation of the reflected ray from the principal axis be α and the incident ray's angle be θ.

$$\begin{aligned} \theta &= \alpha + i + r \\ &= \alpha + 2\beta \end{aligned}$$

Therefore, $$\alpha = \theta - 2\beta \, . \tag{1}$$

By the law of cosines applied to vertex P of triangle PCF, we can write:

$$\left(\frac{R}{2}\right)^2 = R^2 + p^2 - 2R\,p\cos\beta$$
$$\Rightarrow \cos\,\beta = \frac{\frac{3}{4}R^2 + p^2}{2Rp} \, . \tag{2}$$

By the same law applied to vertex F of triangle PCF, we can write:

$R^2 = p^2 + (\frac{R}{2})^2 - 2p\,(\frac{R}{2})\cos(\pi - \theta)$.

Recasting this equation as

$p^2 + (R\cos\theta)\,p - \frac{3R^2}{4} = 0$

gives us

$$p = \frac{R}{2}(-\cos\theta + \sqrt{3 + \cos^2\theta}).$$

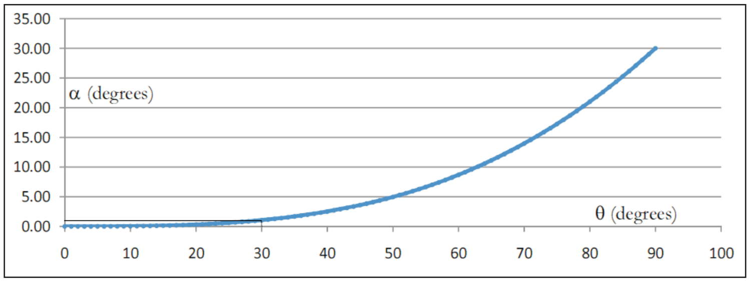


Figure 5. Deviation of reflected ray from axial direction.

Using this along with (1) and (2), we can calculate the deviation α of the reflected ray from the horizontal:

$$\alpha = \theta - 2\cos^{-1}\left(\frac{\frac{3}{4}R^2 + p^2}{2Rp}\right) = \theta - 2\cos^{-1}\left(\frac{\sqrt{(3+\cos^2\theta)}}{2}\right).$$

The behaviour of this function is shown in *Figure* 5 and the behaviour of the direction of the reflected ray arising from this is shown visually in *Figure* 6 for a radius of curvature of 1.

It can be seen from *Figure* 6 that the deviation of the reflected ray from the horizontal starts becoming visually noticeable only after about 30$^\text{o}$ for θ. From *Figure* 5, the value of α for that value of θ is seen to be around 1$^\text{o}$.

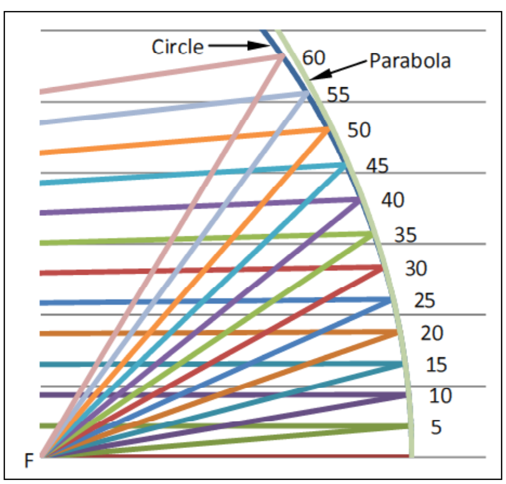


Figure 6. Reflection of rays emanating from the focus F and incident upon a spherically concave mirror at different angles with the principal axis as indicated alongside the ray. The parabola profile is shown for comparison.

So we have determined that spherical mirrors can be considered parallel-beam reflectors only within a range of about 30° (as defined above) from their principal axes. After that the reflected beam deviates significantly from parallelness.

Analytically, the situation is explained by the extent to which a parabola and a circle coincide as one travels away from the principal axis. *Figure* 6 shows both profiles, that of a circle and of a parabola and their divergence from each other is seen to happen around the 30° mark. The parabola compared is one whose focus is also at F in *Figure* 3.

3. Focussing Behaviour

If instead of looking at rays emanating from the focus and studying the extent of their parallelness upon reflection, we look at rays that are parallel to the principal axis when incident upon the mirror (*Figure* 7) and study the extent of their divergence k from the focus (called the 'lateral aberration'), we can quantise the focussing sharpness of spherical mirrors. It is predictably found, based upon the principle of reversibility of light, that narrow beams focus better. This is depicted pictorially and to scale in *Figure* 8.

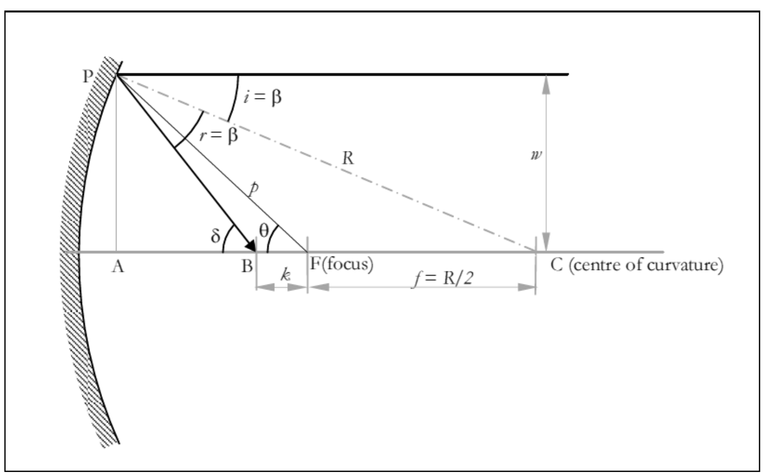


Figure 7. Behaviour of a ray incident parallel to the principal axis and at a distance from it.

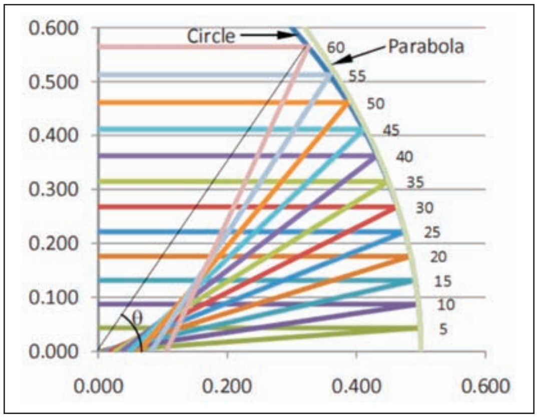


Figure 8. Reflection of rays incident upon a spherically concave mirror parallel to the principal axis. The numbers 5, 10, …, 55, 60 indicate the values of θ for the rays next to which they are written.

In terms of the distance w of the light ray from the principal axis and the radius of curvature R, the lateral aberration k is given by (using distances shown in *Figure* 7),

$$k = l(AF) - l(AB) = \left[\sqrt{R^2 - w^2} - \frac{R}{2}\right] - \frac{w}{\tan\delta}\,.$$

Using

$$\delta = 2\,\beta,\ \text{and}\ \beta\ = \tan^{-1}\left(\frac{w}{\sqrt{R^2 - w^2}}\right)\,,$$

$$\text{and}\ \tan 2x\ =\ 2\ \tan x/(1 - \tan^2 x), \text{we get}$$

$$k = \frac{R^2}{2\sqrt{R^2 - w^2}} - \frac{R}{2}\ . \qquad (3)$$

For a mirror of radius of curvature 1 m and a beam of $\theta = 30^{\circ}$, w comes out to be 0.26 m and this equation yields a value of 1.7 cm for the lateral aberration k. The relationship of k versus θ is graphed in *Figure* 9 and reflects the trend shown in the ray diagram of *Figure* 8.

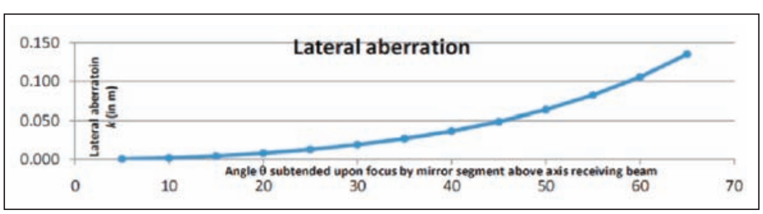


Figure 9. The amount by which a beam misses the focus as a function of its width; this is measured as the angle that half the mirror segment upon which it is incident subtends upon the focus.

The numbers imply that a beam of (semi-)width 26 cm will focus within 1.7 cm of the focal point on a mirror of this curvature. Equation (3) predicts that when the width w of the beam gets larger, the lateral aberration k also increases.

4. Parametrising a Concave Mirror

It is obvious from the above that a spherical mirror gives a parallel beam only for rays incident close to the principal axis. But it is difficult to measure the thirty-degrees-from-axis extent of the mirror. To begin with, this extent is measured from the focal point of the mirror, an imaginary point defined mathematically in terms of the mirror's centre of curvature. The centre of curvature is in turn itself an imaginary point in space and not on some physical body, hence is not directly identifiable. Secondly, the 30° semi-cone is an imaginary cone, not a physical object. There is therefore the need to determine the centre and the angle using physical parameters of the mirror and not imaginary points in space.

The answer to this will also allow us to predict whether a given concave spherical mirror will produce a parallel beam or not.

Assuming that the mirror (*Figure* 10a) has a circular rim, the mirror dimensions that are measurable are the rim diameter D and the 'depth' d of the mirror. The former can be measured using a ruler or vernier calliper and the latter using a vernier height gauge.

Using the equation of a circle through point A in the coordinate frame shown in *Figure* 10b, we can compute the radius of curvature as a function of D and d to be

$$R = \frac{D^2 + 4d^2}{8d}\,. \qquad (4)$$

For the triangle ABC in *Figure* 10b, trigonometry yields

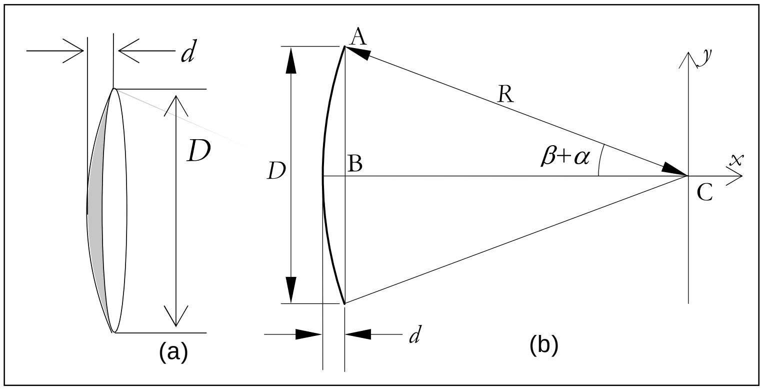


Figure 10. (a) The physically measurable parameters, depth *d* and rim diameter *D*, of a concave mirror. **(b)** The relationship between *d*, *D*, and the radius of curvature *R* of the mirror.

the relation

$$\tan\,(\beta+\alpha) = \frac{D/2}{R-d} = \frac{4Dd}{D^2-4d^2}\ . \qquad (5)$$

As found earlier, when θ is less than 30°, the reflected beam is almost parallel to the principal axis, and α is nearly zero. Then, by (1),

$$\begin{aligned}\theta &= 2\beta,\\ \beta_{\max} &= 0.5\ \theta_{\max}\\ &= 0.5\times 30^{o}\\ &= 15^{o}\ .\end{aligned}$$

Therefore,

$$\begin{aligned}\tan\ \beta &\le \tan 15^{o}\\ &= 0.268\ .\end{aligned}$$

From (5),

$$\tan\ \beta = \frac{4Dd}{D^2-9d^2} \le 0.268\ .$$

This gives

$$\frac{D}{d} \ge 15.5\ . \qquad (6)$$

One can conclude from (6) that when the dimensions of a spherically concave mirror are such that its rim diameter exceeds its depth by more than a factor of about 15, its curvature approximates that of a parabolic reflector in terms of the behaviour of a ray of light passing through its focus or a beam of light incident upon it parallel to its principal axis.

This ratio of D and d when considered along with (4) gives us two relations:

$$d < 0.032\,R, \text{ and } D < 0.51\,R. \qquad (7)$$

These are the limits on the values of d and D for a mirror of given curvature R to behave as a parallel-beam reflector. A to-scale depiction of d and D with respect to R based on the values in equation (7) is illustrated in *Figure* 11.

Corollaries of this reasoning are as follows:

1. Of two mirrors of the same curvature, the smaller one will produce the more parallel/focussed beam. Therefore, cutting a smaller piece out of a given spherical mirror will always give a mirror with a more accurate parallel/focussed beam.

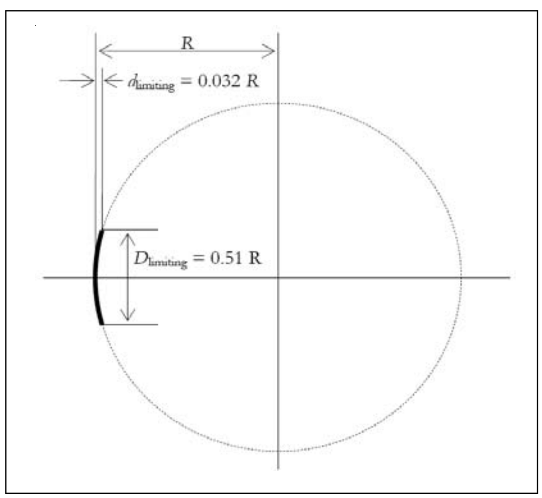


Figure 11. *d* must be less than 0.032 *R* for light passing through the focus to be reflected as a beam of parallel rays. The corresponding limiting value of *D* is 0.51 *R*. When *d* exceeds $d_{limiting}$ (= 0.032 *R)*, the reflected beam ceases to consist of only parallel rays. *D* would also then have exceeded its limiting value of 0.051 *R.* The diagram is to scale.

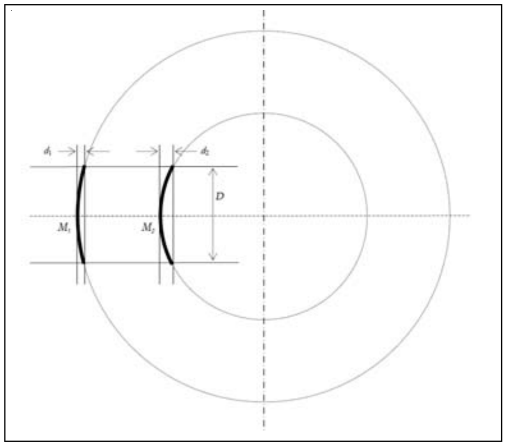


Figure 12. Given two spherical mirrors of the same rim diameter *D*, the one (M_1) with the smaller depth (d_1) can produce a parallel reflected beam better than the one (M_2) with the larger depth (d_2).

2. Of two concave mirrors of the same rim diameter, the one with the smaller depth (and therefore the smaller curvature) will produce the more parallel/focussed beam (see *Figure* 12).

3. Making a larger mirror without compromising on the parallelness of its reflected beam requires even its curvature to be reduced for the parallelness in its reflected beam to persist.

The lateral aberration can be written in terms of the measurable parameters D and d as:

$$k = \frac{d(D^2 + 4d^2)}{2(D^2 - 4d^2)} \; .$$

In a real-world example of a concave mirror of rim diameter 75 mm and depth 2 mm, the results above can be used to derive that the mirror will have a lateral aberration of 1 mm. This is calculated by using (4) to derive R from d and D, and then (3) to calculate the lateral aberration. For the numbers mentioned, we get $R = 352.56$ mm, $w = 37.5$ mm, $\theta = 12.14$ degrees, and the lateral aberration $k = 1.005$ mm. In other words, if turned towards the Sun, it will produce a 'hot spot' of extent 1 mm at its focus. This very small aberration at the focus and the very concentrated hot spot is expected since θ is under 30°.

5. Conclusion

This article analyses the implications of a spherically concave mirror's size upon the mirror's focussing sharpness and upon its ability to produce a parallel beam of light after passing through the focus. It also presents a parametrization of this ability based upon the measurable dimensions – the depth and the rim diameter – of a spherically concave mirror. It finds that a spherical mirror matches a parabolic mirror's precision of focussing only if its rim diameter is larger than about 15 times its depth.

Suggested Reading

[1] http://coraifeartaigh.wordpress.com/2010/03/23/

[2] http://www.phys.ttu.edu/~batcam/Courses/semester%202/Readings/UNIT%2019%20READING%20B%20OPTICS.htm

[3] http://solarcooking.wikia.com/wiki/Parabolic_solar_reflectors

[4] F A Jenkins and H E White, *Fundamentals of Optics*, McGraw-Hill Book Company, 1976.