# Δ Triangular Numbers Δ

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**The triangular numbers, which are numbers associated with certain arrays of dots, were known to the ancient Greeks and viewed by them with reverence. Though possessing a simple definition, they are astonishingly rich in properties of various kinds, ranging from simple relationships between them and the square numbers to very complex relationships involving partitions, modular forms, etc. – topics which belong to advanced mathematics. They also possess many pretty combinatorial properties. In this expository article we survey a few of these properties.**

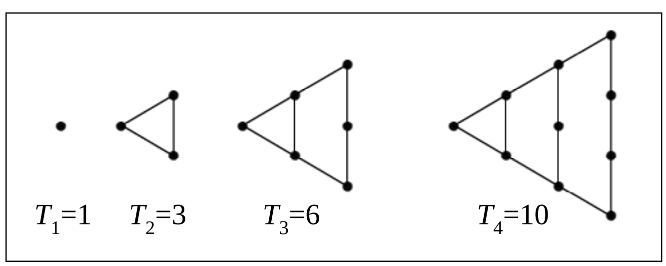
Triangular numbers are numbers associated with triangular arrays of dots. The idea is easier to convey using pictures rather than words; see *Figure* 1. We see from the figure that if $T_n$ denotes the $n$-th triangular number, then $T_1 = 1$, $T_2 = T_1 + 2$, $T_3 = T_2 + 3$, $T_4 = T_3 + 4$, .... Thus $T_n = T_{n-1} + n$, leading to:

$$T_n = 1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}. \qquad (1)$$

The triangular numbers appear in the Pascal's triangle along the third diagonal; in *Figure* 2 they are the numbers in bold type.

**Keywords**
Triangular number, figurate number, rangoli, Brahmagupta–Pell equation, Jacobi triple product identity.

**Figure 1.** The first four triangular numbers.



1
1 1
**1** 2 **1**
1 **3** **3** 1
1 4 **6** 4 1
1 5 **10** **10** 5 1
1 6 **15** 20 **15** 6 1
1 7 **21** 35 35 **21** 7 1
1 8 **28** 56 70 56 **28** 8 1
1 9 **36** 84 126 126 84 **36** 9 1

**Figure 2**. Pascal's triangle.

Triangular numbers were known to the ancient Greeks and were viewed by them with mystical reverence. The triangular number 10 was considered to be a symbol of 'perfection', being the sum of 1 (a point), 2 (a line), 3 (a plane) and 4 (a solid).

Triangular numbers are numbers associated with triangular arrays of dots.

For a sequence defined in such a simple manner, the sequence of triangular numbers abbreviated as $T$-numbers is astonishingly rich in properties. We explore many of them in this article.

## 1. Some Properties of Triangular Numbers

### 1.1 *Triangular Numbers Modulo an Integer*

Here we look at the triangular numbers modulo a positive integer $k$. Reading the triangular numbers modulo 2, we get the following pattern which repeats every 4 steps:

$$1,\ 1,\ 0,\ 0,\ 1,\ 1,\ 0,\ 0,\ 1,\ 1,\ 0,\ 0,\ \ldots.$$

Similarly, reading the numbers modulo 3 gives the following pattern of numbers which repeats every 3 steps:

$$1,\ 0,\ 0,\ 1,\ 0,\ 0,\ 1,\ 0,\ 0,\ 1,\ 0,\ 0,\ \ldots;$$

modulo 4 we get a sequence that repeats every 8 steps:

$$1,\ 3,\ 2,\ 2,\ 3,\ 1,\ 0,\ 0,\ 1,\ 3,\ 2,\ 2,\ 3,\ 1,\ 0,\ 0,\ \ldots;$$

modulo 5 we get a sequence that repeats every 5 steps:

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$$1,\ 3,\ 1,\ 0,\ 0,\ 1,\ 3,\ 1,\ 0,\ 0,\ \ldots,$$

and modulo 6, we get this pattern:

$$1,\ 3,\ 0,\ 4,\ 3,\ 3,\ 4,\ 0,\ 3,\ 1,\ 0,\ 0,\ 1,\ 3,\ 0,\ 4,\ \ldots.$$

After some experimentation, we are able to guess that the $T$-residues mod $k$ repeat every $k$ steps if $k$ is odd, and every $2k$ steps if $k$ is even (see [1]). Once noticed, it is rather easy to prove.

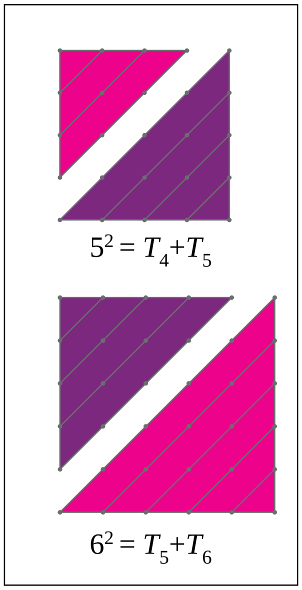



**Figure 3.** Pictorial patterns explaining why the sum of two consecutive $T$-numbers is a perfect square.

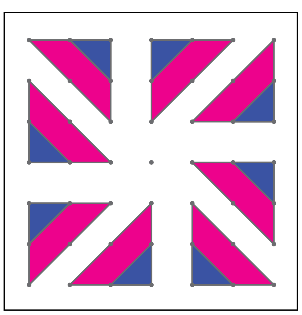
### 1.2 *Triangular Numbers and Squares*

The Greeks knew that the sum of a pair of consecutive triangular numbers is a perfect square. The proof of this using algebra is trivial, but *Figure* 3 shows a nice 'proof without words' which clearly generalizes. The relation in *Figure* 4 ($8T_3 + 1 = 49 = 7^2$) also generalizes in an obvious way; it too was known to the Greeks. Probably hidden inside some well-known 'rangoli patterns' are more such identities for triangular numbers.

### 1.3 *Some Identities*

The following combinatorial interpretation may be given to the $T$-numbers: $T_n$ *is the number of ordered pairs* $(x, y)$*, where* $1 \le x \le y \le n$, $x, y \in \mathbb{N}$. Using this we may prove some identities. For example, the identity

**Figure 4.** A 'rangoli' pattern with 4-fold rotational symmetry; it explains why eight times a $T$-number plus 1 is a perfect square.



$$T_{m+n} = T_m + T_n + mn, \qquad (2)$$

is proved as follows: $T_{m+n}$ counts the number of pairs $(x, y)$ with $1 \le x \le y \le m+n$. We partition the collection of all such pairs into three classes: (i) pairs $(x, y)$ with $1 \le x \le y \le m$; (ii) pairs $(x, y)$ with $m+1 \le x \le y \le m+n$; and (iii) pairs $(x, y)$ with $1 \le x \le m < y \le m+n$. Counting these three classes separately, we get (2).

Relation (2) has the following unexpected consequence. Let a heap of beads (any number) be placed on a table. It is divided it into two heaps in any way one pleases, and the product of the sizes of the two new heaps is recorded. This step is then iterated: each of the resulting heaps is subdivided into two new heaps, etc. After some steps each heap will have just one bead and no further division is possible. Now let all the products recorded be summed. It will be found that whichever way one does it, the total obtained in the end is the same; it depends only on the number of beads in the starting heap. Try it out, and explain how it follows from (2).

There exist many connections within the sequence of *T*-numbers, and between the *T*-numbers and the squares.

Another such relation is:

$$T_{mn} = T_m T_n + T_{m-1} T_{n-1}; \qquad (3)$$

it has the following corollary, which is obtained by putting $m = n$:

$$T_{n^2} = (T_n)^2 + (T_{n-1})^2. \qquad (4)$$

Thus, there exist many connections within the sequence of $T$-numbers, and between the $T$-numbers and the squares. For more such examples see [2]. These properties, once spotted, are easy to prove, but the greater challenge lies in finding them.

### 1.4 *Relations Among the Triangular Numbers*

Here are some more relations among the triangular numbers which are easy to verify algebraically:

If $n$ is a triangular number, then so is $9n + 1$. (5)

If $n$ is a triangular number, then so is $25n + 3$. (6)

For example, for (5): if $n = \frac{1}{2}k(k+1)$, then $9n + 1 = \frac{9}{2}k(k+1) + 1 = \frac{1}{2}(3k+1)(3k+2)$. Euler posed (and answered) the following problem:

*Find all integer pairs $(r, s)$ which satisfy the following condition: If $n$ is a triangular number, then so is $rn + s$.*

Going deeper, we get the following result whose proof we leave as an exercise:

It is of interest to know which triangular numbers are themselves squares.

**Theorem 1.1.** *The integer pairs* $(r, s)$ *for which the property 'n triangular implies* $rn + s$ *triangular' holds are those for which* $r$ *is an odd square, and* $s = \frac{1}{8}(r-1)$.

Fittingly, each such $s$ is a triangular number. For more such relations, see [3].

## 2. Which Triangular Numbers are Squares?

Having seen some connections between triangular numbers and squares, it is of interest to know which triangular numbers are themselves squares. This question has been well-studied, and one has the following striking result originally proved by Euler in 1730.

See *Resonance* issues on Euler, Gauss, Brahmagupta: Vol.2, Nos 5 & 6, 1997; Vol.17, No.3, 2012.

**Theorem 2.1.** *A triangular number* $T_n$ *is a square if and only if* $n$ *has the form*

$$n = \frac{\left(\sqrt{2}+1\right)^{2k} + \left(\sqrt{2}-1\right)^{2k} - 2}{4},$$

*where* $k$ *is an integer. In particular, there exist infinitely many triangular numbers which are squares.*

A quick proof goes as follows: if $\frac{1}{2}n(n+1) = m^2$ for some $m$, then $(2n+1)^2 - 8m^2 = 1$. By a suitable change of variables ($x = 2n+1$, $y = 2n$), we get the Brahmagupta–Pell equation which has both a rich history and a rich theory behind it:

$$x^2 - 2y^2 = 1.$$

Solving this we get the first few values of $n$ as 1, 8, 49 and 288, with corresponding $T$-values 1, 36, $1225 = 35^2$, and $41616 = 204^2$.

There are many ways of arriving at the general solution of the Brahmagupta–Pell equation. One approach is to consider the number field $\mathbb{Q}(\sqrt{2})$ and to define a function $f$ on this field as follows: if $x = a + b\sqrt{2}$ where $a, b \in \mathbb{Q}$, then $f(x) = a^2 - 2b^2$. We seek all elements of $\mathbb{Z}(\sqrt{2})$ with unit value for $f$. It is easy to check that $f$ is multiplicative; i.e., if $x, y \in \mathbb{Q}(\sqrt{2})$, then $f(xy) = f(x)f(y)$. Let

The Brahmagupta–Pell equation has both a rich history and a rich theory behind it.

$u = 1 + \sqrt{2}$. Since $f(u) = -1$, it follows that $f(u^{2k}) = 1$ for any integer $k$. Following through with this observation we get the 'if' part of the above result. The 'only if' part is obtained by a descent procedure, in which from any given solution we construct a smaller one by multiplying by $v = 1 - \sqrt{2}$.

The equation in Section 2.2 is more difficult to solve than the Brahmagupta–Pell equation; it involves elliptic curves.

### 2.2 *. . . and Cubes?*

A natural question to ask now is: do there exist triangular numbers that are perfect cubes? If there are such numbers, then one can simplify as above to get an equation of the form:

$$(2n + 1)^2 - (2m)^3 = 1.$$

This equation is more difficult to solve than the Brahmagupta–Pell equation; indeed, it involves questions about elliptic curves. It leads naturally to the famous Catalan conjecture (posed in 1844, and proved in 2002 by Preda Mihăilescu) which makes an assertion about pairs of perfect powers that differ by 1. The unexpectedly simple answer to this (that the only non-trivial instance is $3^2 - 2^3 = 1$) tells us that the only triangular number which is a cube is $T_1 = 1$.

## 3. Triangular Numbers and Sums of Squares

Carl Friedrich Gauss (1777–1855), one of history's most influential mathematicians, discovered that every positive integer can be written as a sum of at most three triangular numbers; on 10 July 1796 he wrote these famous words in his diary:

$$\text{EΥPHKA!} \quad \text{num} = \Delta + \Delta + \Delta.$$

Every positive integer can be written as a sum of at most three triangular numbers.

However there is no simple proof of this statement, which is sometimes called the 'Eureka Theorem'.

It is easy to prove the following characterization of triangular numbers: *k is a triangular number if and only* *if* $8k+1$ *is a square.* For:

$$n = \frac{x(x+1)}{2} \iff 8n = 4x^2 + 4x \iff 8n+1 = (2x+1)^2.$$

More generally we have (see [4], Proposition 2): *A positive number* $n$ *is a sum of* $k$ *triangular numbers if and only if* $8n+k$ *is a sum of* $k$ *odd squares.* For the particular case $k=2$, we have the following result (see [5]):

**Theorem 3.1.** *A natural number* $n$ *is a sum of two triangular numbers if and only if in the prime factorization of* $4n+1$*, every prime factor* $p \equiv 3 \pmod 4$ *occurs with an even exponent.*

We justify the 'only if' part as follows: if $n = \frac{1}{2}x(x+1)$ then $4n+1 = x^2 + (x+1)^2$ is a sum of two squares, and the laws concerning such numbers are well known: *A positive integer* $N$ *is a sum of two squares if and only if every prime* $p \equiv 3 \pmod 4$ *which divides* $N$ *does so to an even power.* Hence the stated result. The 'if' part, as in most such results, is more challenging.

Gauss's Eureka theorem shows that every positive integer is a sum of at most three triangular numbers. Given a positive number, we may ask: how many ways are there to write it as a sum of triangular numbers? For example, the (unlucky!) number 13 can be written as such a sum in two ways: $13 = 3+10 = 1+6+6$. This question is difficult to answer, but there are nice results about the number of representations of a given number as a sum of *two* triangular numbers, expressed in terms of certain divisor functions. Such a relation could have been anticipated: such formulae are already known in the context of expressing positive integers as sums of squares, and we also know that triangular numbers are closely related to squares. (The problem of number of representations of a given number as sums of squares was first studied by Greek mathematicians and has interesting connections to the study of the geometry of

Gauss's Eureka theorem shows that every positive integer is a sum of at most three triangular numbers.

lattice points in a plane. For an interesting account see [6].)

We now fix some notation and recall some relevant results from [7]. Let $r_2(n)$ be the number of representations of $n$ as a sum of two squares, and let $t_2(n)$ denote the number of representations of $n$ as a sum of two triangular numbers:

$$r_2(n) = \left|\{(x,y) \in \mathbb{Z}^2 \ : \ x^2 + y^2 = n\}\right|,$$
$$t_2(n) = \left|\{(x,y) \in \mathbb{Z}^2 \ : \ x(x+1) + y(y+1) = 2n\}\right|.$$

Further, let $d_i(n)$ be the number of divisors of $n$ which are of the form $i \pmod 4$. Then one has the following nice connections (see [2]):

**Theorem 3.2.**

$$t_2(n) = d_1(4n+1) - d_3(4n+1), \quad r_2(4n+1) = 4t_2(n).$$

One may ask, more generally, about how many representations $n$ has as a sum of $k$ triangular numbers. The analysis of this question is difficult and reveals connections with the theory of modular forms, an extremely active area of current research. For details see [4].

One of the results used in the analysis of this problem is Jacobi's well-known *triple product identity* (see [8], page 238). Another theorem of Jacobi which arises in the context of partitions of numbers (see [9], Theorem 357) and involves the triangular numbers is:

$$\prod_{n=1}^{\infty}(1-x^n)^3 = \sum_{m=0}^{\infty}(-1)^m(2m+1)x^{m(m+1)/2}.$$

The relation between triangular numbers and partitions of natural numbers is very deep, and we offer the following result as another example (see [9], Theorem 355). We know that the generating function

$$\frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}$$

The relation between triangular numbers and partitions of natural numbers is very deep.

enumerates the partitions of natural numbers into odd parts, whereas

$$\frac{1}{(1-x^2)(1-x^4)(1-x^6)\cdots}$$

enumerates the partitions into even parts. The astonishing formula which ties these two generating functions to triangular numbers reads:

$$\frac{(1-x^2)(1-x^4)(1-x^6)\cdots}{(1-x)(1-x^3)(1-x^5)\cdots} = \sum_{n=0}^{\infty} x^{n(n+1)/2}.$$

Thus, these simple looking numbers take us a long way into the current areas of research in number theory.

## 4. Other Figurate Numbers

Questions similar to those asked above can be asked for numbers arising from geometrical figures like squares, pentagons, hexagons, etc.; see *Figure* 5. Numbers related to such figures are called *figurate numbers* and they exhibit many interesting properties. Thus, one can define pentagonal numbers, hexagonal numbers, heptagonal numbers, and so on. The formula for the $r$-th $k$-gonal number is:

$$\frac{r\big((k-2)r-(k-4)\big)}{2}.$$

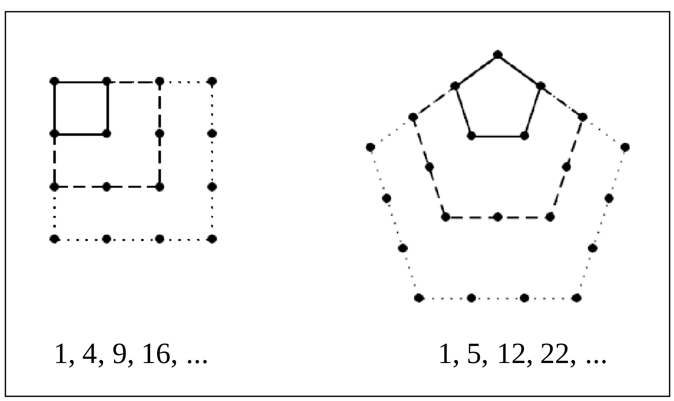



**Figure 5.** Square numbers and pentagonal numbers.

We refer the reader to an article by Richard K Guy [10] for a list of some unsolved problems associated with these families of numbers. There are many interesting relations between these numbers and we leave it to the reader to find algebraic and pictorial proofs for them.

## Acknowledgements

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