Classroom

In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

Priyanka deSouza[1] and
Vijay A Singh[2]
[1]Department of Energy Science and Systems Engineering, IIT Bombay, Mumbai 400 076, India.
Email: priyankadesouza@gmail.com

[2] Homi Bhabha Centre for Science Education (TIFR)
V N Purav Marg, Mankhurd
Mumbai 400 088, India.
Email: vsingh@hbcse.tifr.res.in

Keywords
100 m dash, kinematic models, universality.

Simple Models for the 100 Meter Dash

We propose simple models for the acceleration of top athletes in the 100 m dash. Our approach is empirical. We assume that the athlete begins with a high acceleration that attenuates exponentially. We also propose an alternative three stage linear model for the acceleration. The models successfully reproduce the observed run profile of the top three finalists in the 100 m dash at the 2009 World Championship in Berlin. We find the exponential model more attractive since it provides a natural time scale namely the decay constant for the acceleration and has fewer adjustable parameters. A common finding from both models is that a high acceleration cannot be sustained. Our findings suggest that the models are universal: the kinematics of several top athletes in the 100 m dash are similar. We speculate on the possibility of a sub 9.5 s run for Bolt. Our exponential model could provide insights for the construction of a dynamical model. Our approach is pedagogical and can serve as an example of model building in physics.

1. Introduction

Few human events are as spectacular as the 100 m dash. At the 2009 World Championship in Berlin, Usain Bolt of Jamaica electrified the audience with a 100 m dash in a record breaking 9.58 s. A year earlier at the Olympics in Beijing, Bolt had achieved a time of 9.69 s. Our objective in this study is to construct simple models of the acceleration. We explore some consequences of our models. We also speculate on how one can modulate the kinematic parameters to produce a sub 9.5 s run. Our exercise could be used by teachers to make students gain an appreciation of kinematics and also to encourage them to construct models in physics.

We note that there have been studies on sprinting by physicists. In the widely cited optimization approach by Keller [1] the resistive force was taken to be velocity dependent. Subsequently several workers have adopted this model [2, 3, 4]. Murieka has also proposed a model for the 100m dash [5]. A detailed reference list can be found in the work of Heck and Ellermeijer [4]. But as Eriksen has pointed out, most of them have attempted to construct a dynamical model [6]. Our perspective is empirical and confined to constructing a kinematic model for sprinting. It may lead to the construction of an appropriate dynamical model, but that remains an open question.

In Section 2, we outline two simple models for the acceleration. In Section 3 we employ it to explain the run profiles of Usain Bolt, Tyson Gay and Asafa Powell who were the top three finalists in the 2009 World Championship held in Berlin. Section 4 constitutes the discussion.

2. Models

At this point we suggest that the reader watch the 100 m dash a few times to gain an appreciation for the models we are going to propose [7]. Bolt is our object of study, but we found that all the finalists display a similar run pattern.

2.1 *Exponential Model*

The simplest model is to assume that the athlete begins with a high acceleration which then attenuates exponentially:

$$a(t) \;=\; a_0 e^{-t/\tau}, \qquad 0 \;<\; t \;<\; t_f\,. \qquad (1)$$

Here t_f is the time of completion of the race. Given initial conditions that velocity $v(0) = 0$ and distance $x(0) = 0$, one can easily show that

$$v(t) = -\tau a_0 e^{-t/\tau} + \tau a_0\,, \qquad (2)$$

$$x(t) = \tau^2 a_0 e^{-t/\tau} - \tau^2 a_0 + \tau a_0 t\,. \qquad (3)$$

For the sake of illustration let us take the initial acceleration $a_0 = 8\,\mathrm{m{\cdot}s^{-2}}$ and τ =1.5 s. A simple calculation then tells us that the terminal velocity $a_0\tau$ is 12m·s^{-1} and the athlete can complete the race in 9.82 s. In this model the constants a_0 and τ assume special significance. The initial burst of acceleration of the athlete is represented by a_0, and τ is the time decay constant which indicates how long an athlete is able to sustain the acceleration. We shall see later that all three athletes have almost similar decay constants of approximately 1.5 s.

2.2 *Linear Model*

Our second model is based on the analysis of Bolt's run by Eriksen *et al* [6]. Using video clips of the run, the authors had fit the distance–time $x(t)$ data to a spline. By differentiating they had obtained Bolt's velocity $v(t)$ and acceleration profiles $a(t)$. Based on the data provided in their *Table* 1, we have repeated this exercise and our acceleration profile $a(t)$ shown in *Figure* 1 is similar but of a somewhat poorer quality.

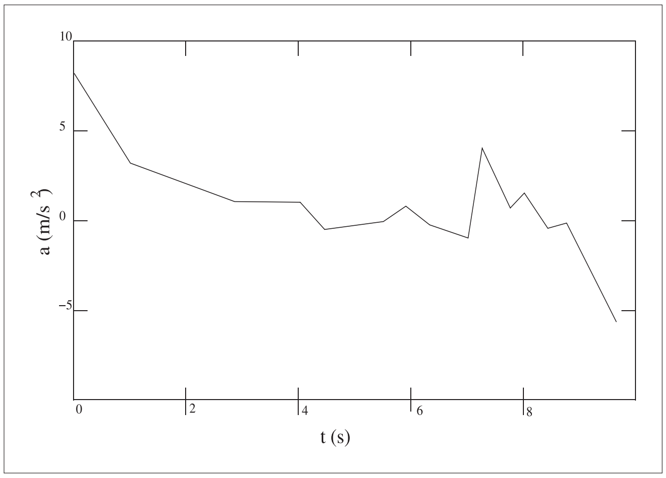

Figure 1. A reconstruction of the acceleration of Usain Bolt in the 2008 Beijing Olympics from the data provided by Ericksen *et al* [5] for his 100 m dash.

We now examine the reconstructed acceleration curve in *Figure* 1. It has some noisy features. Nevertheless its broad outline can be abstracted. After a number of trials we settled on the following linear fits.

The high acceleration followed by a rapid fall in the first third of the race is modeled linearly. We suggest

$$a(t) \;=\; a_0 - a_1 t\,, \qquad 0 \;<\; t \;<\; t_1;\; t_1 = a_0/a_1\,. \quad (4)$$

The next part of the figure is oscillatory. These oscillations however are slightly offset from $a = 0$ on the negative side. We model it very simply as

$$a(t) = 0\;, \qquad t_1 \;<\; t \;<\; t_2\;. \qquad (5)$$

The final third of the race posed some problems. There is a spike in the acceleration and then a decline into deceleration with actual slowing down. This appears counter intuitive. The athlete is expected to clinch the race with a strong burst of speed. But there is some support for negative acceleration [6, 8]. For simplicity we assume that the spike is instantaneous and linearize the falling acceleration by:

$$a(t) \;=\; a_2 - a_3(t - t_2)\;, \qquad t_2 \;<\; t \;<\; t_3\;, \quad (6)$$

where t_3 is the time taken for the completion of the race.

To be concrete let us take $a_0 = 6$ m·s^{-2} and $a_1 = 1.5$ m·s^{-3}. Given the initial condition $x(0) = v(0) = 0$ this shows that the first stage of the race takes 4 s ($t_1 = 4$ s). We suggest that $t_2 = 7$ s and for definiteness we take $a_2 = 2.55$ m·s^2 and $a_3 = 8/3$ m·s^3. A straightforward calculation yields a critical time $t_c = 8.125$ s, where $a(t_c) = 0$. For t > t_c, the athlete actually slows down. With these values of the parameters one can also show that the athlete traverses 32 m in the first stage, another 36 m in the second stage and completes the race in 9.58 s.

3. Results

The above models for the acceleration can be tested. At the Berlin World Championship in August 2009, the Scientific Research Project group (SRP) carried out a bio-mechanical observation and analysis [9]. The split times and analyses are also available on another website [10]. Columns 1 and 2 of *Table* 1 list the data for Usain Bolt who won the race in an astonishing 9.58 s. Column 3 is our predicted $x_B(t)$ based on the exponential model

Table 1. A comparison of the observed *x*(*t*) and calculated distances of Bolt $x_B(t)$, Powell $x_P(t)$ and Gay x_G(t) for the 100 m dash in the Berlin World Championship 2009. The times are taken from [8, 9]. The distance is calculated based on the exponential model proposed in Section 2.1 with the numerical values of the parameters given in *Table* 2. Data for Gay at 10, 50, 70 and 90 m was not available.

	Usain Bolt		Asafa Powell		Tyson Gay	
$x(t)$	t_B	$x_B(t)$	t_P	$x_P(t)$	t_G	$x_G(t)$
0	0	0	0	0	0	0
10	1.89	10.271	1.87	10.133	–	–
20	2.88	20.069	2.90	20.183	2.92	20.087
30	3.78	30.027	3.82	30.152	3.83	30.021
40	4.64	40.038	4.70	40.121	4.70	40.009
50	5.47	49.952	5.55	49.960	–	–
60	6.29	59.883	6.39	59.789	6.39	60.057
70	7.1	69.769	7.23	69.677	–	–
80	7.92	79 .820	8.08	79.714	8.02	79.719
90	8.75	90.020	8.94	89.886	–	–
100	9.58	100.234	9.84	100.541	9.71	100.214

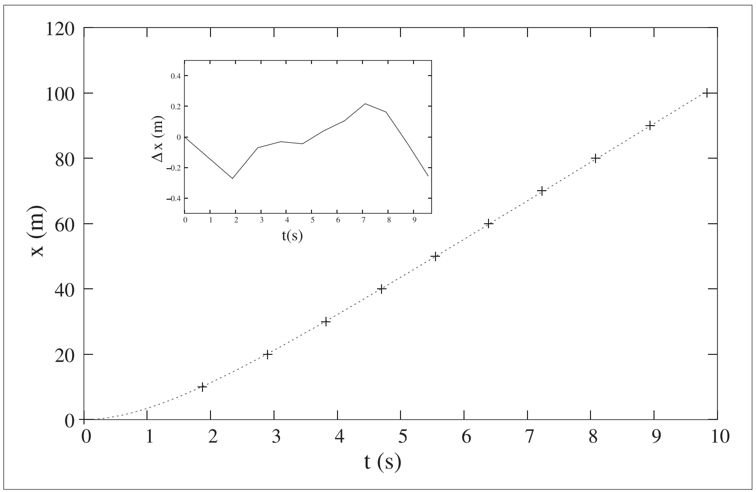

Figure 2. Plot of the observed data points (crosses) and calculated distance (dotted line) traversed by Usain Bolt in the 2009 Berlin World Championship. The calculated distance is based on the exponential model described in Section 2 with the parameters for Bolt given in *Table* 2. The inset which shows the difference between the observed distance and the calculated distance of Usain Bolt highlights the good agreement between theory and observation.

in Section IIA with the numerical values of the parameters given in *Table* 2. The agreement is up to ± 0.3 m. This is heartening since the model for acceleration we proposed was deliberately kept simple.

Column 5 of *Table* 1 represents our calculations for Asafa Powell who stood third in the race. Column 7 depicts the corresponding calculations for Tyson Gay who was the runner up and for whom the data is available at intervals of 20m. The agreement for Powell is up to ± 0.55 m and for Tyson it is ± 0.3 m.

In *Figure* 2 we depict the run profile for Usain Bolt. The inset depicts the difference between the observed and calculated values. In *Figure* 3 we depict the run profile of Powell for whom the data was available at every 10 m [9, 10]. The inset shows a similar figure for Tyson Gay who was the runnerup. The figures provide a visual confirmation of the excellent agreement noted in *Table* 1.

We note that in the exponential model, the calculated velocities decay exponentially to a terminal value $(a_0\tau)$. From *Table* 2 we have that the terminal velocities for Bolt, Gay and Powell are $12.33\,\text{m}\cdot\text{s}^{-1}$, $12.15\,\text{m}\cdot\text{s}^{-1}$, and $11.98\ \text{m}\cdot\text{s}^{-1}$ respectively. The terminal velocity of the

Table 2. The values of the acceleration parameters based on the exponential model for the athletes' 100 m dash in the Berlin World Championship 2009. See Section 2.1 for the relevant equations.

	a_0	τ
Bolt	8.491	1.452
Gay	8.276	1.469
Powell	8.733	1.357

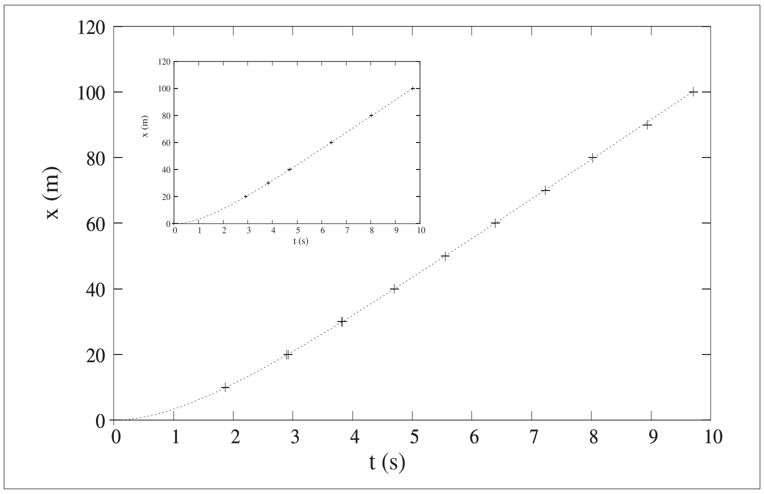

Figure 3. Plot of the observed data points (crosses) and calculated distance (dotted line) traversed by Asafa Powell in the 2009 Berlin World Championship. The calculated distance is based on the exponential model described in Section 2 with the parameters for Powell given in *Table* 2. The inset is the observed data points and the calculated distances for Tyson Gay.

three athletes tracks their performance with Bolt being the fastest and Gay and Powell following.

In *Table* 3 we compare the run profiles for the three athletes with the linear model of Section 2.2. The parameters are given in *Table* 4 and were obtained using regression analysis as well as some trial and error [12]. The agreement for Bolt is up to ± 1.1 m, for Powell up to ± 0.9 m and for Tyson it is ± 1.15 m. This is satisfactory but not as good as the exponential model.

Table 3. A comparison of the observed *x*(*t*) and calculated distances of Bolt $x_B(t)$, Powell $x_P(t)$ and Gay $x_G(t)$ for the 100 m dash in the Berlin World Championship 2009. The times are taken from [8, 9]. The distance is calculated based on the linear model proposed in Section 2.2 with the numerical values of the parameters given in *Table* 4.

	Usain Bolt		Asafa Powell		Tyson Gay	
$x(t)$	t_B	$x_B(t)$	t_P	$x_P(t)$	t_G	$x_G(t)$
0	0	0	0	0	0	0
10	1.89	9.03	1.87	9.12	–	–
20	2.88	18.91	2.90	19.49	2.92	18.85
30	3.78	29.36	3.82	29.99	3.83	29.33
40	4.64	39.68	4.70	40.02	4.70	39.71
50	5.47	49.64	5.55	49.72	–	–
60	6.29	59.48	6.39	59.30	6.39	59.91
70	7.1	69.21	7.23	68.95	–	–
80	7.92	79.77	8.08	79.68	8.02	80.36
90	8.75	90.52	8.94	90.51	–	–
100	9.58	99.81	9.84	99.97	9.71	99.72

	a_0	a_1	a_2	a_3	t_1	t_2	t_3
Bolt	6	1.5	2.55	8/3	4.0	7.0	9.58
Gay	5.785	1.4	2.53	8/3	4.13	6.90	9.71
Powell	6.28	1.7	2.85	8/3	3.69	7.0	9.84

Table 4. The values of the acceleration parameters based on our linear model for the athletes' 100 m dash in the Berlin World Championship 2009. See Section 2.2 for the relevant equations.

The parameters of the exponential model in *Table* 2 were obtained using a Levenburg–Marquardt fit coupled with some trial and error [11]. A regression analysis coupled with trial and error was used to obtain the parameters for the linear model in *Table* 4 [12]

We now comment on the possibility of errors. The times quoted by the World Championship officials and the Scientific Research Project (SRP) are up to the second decimal place. Hence the accuracy may be presumed to be at ± 0.01 s. We point out that in their data display, the SRP revised their times for Bolt at three positions $(20, 30$ and $60\,\text{m})$ by as much as 0.02 s. $(2.88, 3.78$ and 6.29 s were revised upwards to $2.89, 3.79$ and 6.31 s respectively). A similar revision for Asafa Powell took place at five positions $(20, 30, 40, 60, 90\,\text{m})$ by as much as 0.03 s. $(2.90, 3.82, 4.70, 6.39, 8.08$ s to $2.91, 3.83, 4.71, 6.42, 8.10$ s). Thus it appears that the error margin of the intermediate distances could be 0.03 s, which at $10\,\text{m} \cdot \text{s}^{-1}$ translates to a distance error of 0.3 m. We also note that our predictions are not too off, being at most $\pm$ 0.55m. The sprinter is not a rigid body and we doubt whether a higher accuracy for intermediate distances can ever be achieved. Maximum running speed measurements is not a simple exercise [13].

4. Discussion

Although Usain Bolt is the clear winner we find that the acceleration profiles of Tyson Gay and the other participating athletes are similar to that of Bolt's. We note from *Table* 2 that both the initial acceleration a_0 and the time constant τ of Bolt is intermediate between

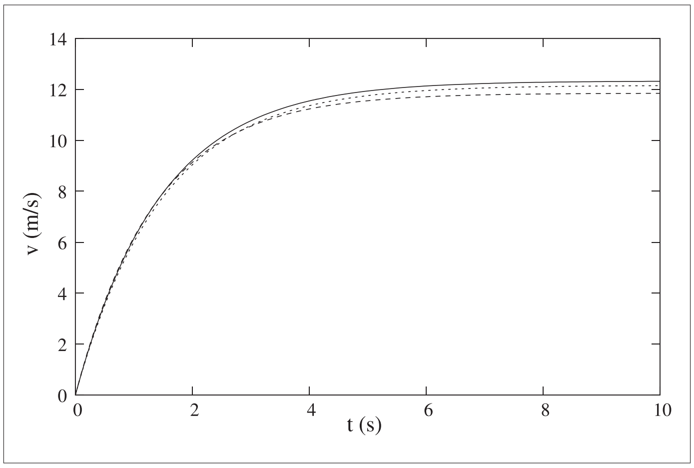

Figure 4. Plot of the calculated speeds of Bolt (solid line), Gay (dotted line) and Powell (dashed line) versus time. The calculated speeds are based on the exponential model described in Section 2 with the parameters given in *Table* 2. The three speed profiles are similar.

those of Gay and Powell. However the values are close for all three athletes ($a_0 \approx 8.5\,\mathrm{m} \cdot \mathrm{s}^{-2}$ and $\tau \approx 1.5\,\mathrm{s}$). Wagner has analysed the data for sprinters Carl Lewis and Ben Johnson and arrived at similar time constants [3]. Thus there is a 'universality' underlying the efforts of the top runners. This is visually reinforced by the speed profiles of the three top athletes in *Figure* 4. Bolt is better, but only incrementally so. We note however that there exist other studies which do not take this position [14]. A useful exercise which we may undertake in the future would be to look at detailed run profiles of different atheletes over a period of several years [15].

We have presented two models for sprinting. The observable is $x(t)$ and hypothesizing $a(t)$ is an example of the 'inverse problem' in science. For example, in molecular physics one observes the energy spectrum and one needs to guess the atomic species comprising the molecule. Often the inverse problem does not have a unique solution and while we have presented two models, we have developed a number of others which we are not discussing here.

The linear model is a 5-parameter exercise, while the exponential model has two parameters, the initial acceleration and the time constant. On this count alone the exponential model appears attractive. While both models fit the run profiles of the three athletes very well, the exponential model is a shade better. Further the exponential model has a decay constant for the effort. This could be linked to the energy expenditure and fatigue. We also note that in both models a high acceleration is followed by a rapid decay.

The power output $P(t)$ of the athlete is the product of the force and the speed. We thus have from (1) and (2) that

$$P(t) = ma_0^2 \tau e^{-t/\tau}(1 - e^{-t/\tau}), \tag{7}$$

where m is the mass of the athlete. We note that the power output is zero at $t = 0$, the start of the race. Since the power is the time derivative of the energy it follows that the maximum energy is required at the start of the race. Further the power peaks at $\tau \ln 2$ as can be verified by differentiating the above expression for $P(t)$ and setting the resulting equation to zero. This is approximately 1 s after the race has begun (see *Table* 2 for τ). While we need not take this exercise at face value it does suggest that power considerations should be a starting point for the construction of a dynamical theory[1]. This is in contrast to the constant propulsive force model with velocity-dependent retardation and its modifications [1, 16]. A careful analysis using inputs from physiology and bio-mechanics may hold the key to such a theory [17, 18].

The above comparisons between the three runners lets us to speculate on the possibility of Bolt completing the dash in less than 9.5 s. We have tried to change the following parameters in his run profile: (i) We see that Powell's initial acceleration is higher than that of Bolt's. If Bolt had this inital acceleration then we find that he would have completed the race in 9.33 s. (ii) On the other hand, we note that Gay is able to sustain his acceleration a trifle longer than Bolt. If Bolt had the same

[1] Classical mechanics has two branches: kinematics and dynamics. The part which deals with the motion of the object without consideration of the causes leading to the motion, such as Galileo's laws of falling bodies, is called kinematics. The part which offers causal explanation of the motion, i.e., the study of forces on an object, along with its description is called dynamics. In dynamical theories we identify the forces and apply Newton's laws of motion.

time constant as Gay, then our calculations show that he would complete the race in 9.48 s. (iii) Assuming the most favourable circumstances, if Bolt had Powell's initial acceleration and Gay's time constant, then we find that he would complete the race in an astonishing 9.26 s. It must be borne in mind that this and the sub 9.5 s timings mentioned above are accurate within ± 0.1 s. (iv) It has been observed that Usain Bolt has a slow reaction time. His reaction time was the slowest in the Beijing Olympics [6]. Improving the reaction time would give him an advantage of at most 0.03 s thus enabling him to complete the race in 9.55 s. We have ignored wind speed considerations.

Acknowledgment

This work was supported by the National Initiative on Undergraduate Sciences (NIUS) undertaken by the Homi Bhabha Centre for Science Education (Tata Institute of Fundamental Research), Mumbai, India. We are grateful to Praveen Pathak (HBCSE) for several useful discussions.

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