'Length' at Length

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This is the story of length, one of the simplest concepts. Starting with the concept of length for intervals, we examine the collection of sets for which we can associate length.

1. Length

We shall be dealing, in this article, with subsets of the interval $[0, 1]$. Of course, one can consider the real number system, but then one has to deal with intervals of infinite length, and this needs extra care.

What is the mathematical concept of length? If you consider the interval $I = [0.3, 0.5]$, that is, the set of all numbers which are at least 0.3 and at most 0.5, then the length of the interval I equals $0.5 - 0.3 = 0.2$. More generally an interval $[a, b]$ has length $b - a$. This is expressed by writing $L[a, b] = b - a$. Recall that $[a, b]$ is the set of all numbers x such that $a \leq x \leq b$.

What about singleton sets? Following the maxim 'points have no length', it is intuitively felt that they have length zero. Of course, you could say that singleton set $A = \{0.8\}$ is just the interval $I = [0.8, 0.8]$; the formula above holds for such intervals too and hence A has length $0.8 - 0.8 = 0$. You can also justify this intuitive feeling as follows. Clearly, $A \subset [0.8 - 0.01,\ 0.8 + 0.01]$. It is reasonable to believe that length of A cannot be more than the length of the interval $[0.8 - 0.01,\ 0.8 + 0.01]$ which equals 0.02. Since you can replace the number 0.01 by any smaller positive number, you see that length of A must be smaller than every strictly positive number and hence $L(A) = 0$. Thus you should accept that length of a singleton set is zero.

Keywords
Translation invariance, axiom of choice, continuum hypothesis, axiom of determinateness.

Let us recall that the interval $(a, b]$ is the set of all numbers which are strictly larger than a, but at most equal to b. Thus the number a is not an element of this set $(a, b]$ but the number b is an element. Similarly (a, b) denotes the set of all numbers which are strictly between a and b. Thus this set does not include the numbers a and b. Since the intervals $[0.3, 0.5]$ and $(0.3, 0.5]$ differ by only one point which has length zero, it is natural to feel that $L(0.3, 0.5] = 0.2$ and similar reasoning tells you that $L(a, b) = b - a$. Thus for an interval, whether it be (a, b) or $[a, b)$ or $(a, b]$ or $[a, b]$, its length is the difference betweeen the end points, namely, $b - a$.

If you consider the two intervals $(0.3, 0.5]$ and $(0.5, 0.8]$, their union is the interval $(0.3, 0.8]$ and as you could see length adds up. Even if the intervals are not contiguous, it is natural to feel that this should be true. In other words, for two disjoint intervals I_1 and I_2, we should have $L(I_1 \cup I_2) = L(I_1) + L(I_2)$. For example if $I_1 = (0.3, 0.5]$ and $I_2 = (0.7, 0.9]$ then $L(I_1 \cup I_2) = 0.4$. If it hurts you to think of length for the union of two non-contiguous intervals, you may rename it as 'total length' or 'size', but we continue to use the word length. In particular, if A is a set consisting of two points, more generally finitely many points, then its length should be zero.

Here is another property of length. If you take a sequence of intervals $\{I_n; n \geq 1\}$ which are pairwise disjoint and if their union is again an interval I, then length adds up, that is, $L(I) = \sum L(I_n)$. In other words length adds up, not only for finitely many disjoint intervals, but also for countably many disjoint intervals. This fact seems to have been noticed [1] by the German Mathematician Georg Cantor around 1884. Here is the reason why this holds.

Length adds up not only for finitely many disjoint intervals but also for countably many.

For any integer $k \geq 1$, you can re-order the disjoint intervals $I_1, \cdots, I_k$ so that each interval is to the right

Countable sets have length zero.

of the previous interval and then add their lengths to see $\sum_1^k L(I_n) \leq L(I)$. Since k is arbitrary, you conclude $\sum L(I_n) \leq L(I)$. To prove $\sum L(I_n) \geq L(I)$ is tricky. One uses an ϵ-argument which involves reducing I 'a little bit' to get a closed interval and enlarging the interval I_n 'a little bit' (by $\epsilon/2^n$) to get an open interval and using a compactness argument. You can find the details in any of the references [2–4]. The complication in the proof is due to the following reason.. Unfortunately, when you have infinitely many intervals, you may not be able to reorder them as one sequence $J_1, J_2, \cdots$, so that each interval is to the right of the previous one (or each interval is to the left of the previous interval). Once you realize that this can happen, it is not difficult to make examples. We shall not pause to give an example, since it will divert attention from the main intent of our discussion.

Let $\mathcal{C}$ be the class of sets which are finite or countable unions of pairwise disjoint intervals. If C is the union of disjoint intervals $\{I_n; n \geq 1\}$, then simply declare $L(C)$ as $\sum L(I_n)$. Let us note that $(a, a]$ is the empty set and $[a, a]$ is the singleton set $\{a\}$ and hence these sets are in $\mathcal{C}$. Thus countable sets are in the collection $\mathcal{C}$ and their length equals zero.

Since C may be represented as union of disjoint intervals in more than one way, you need to show that this prescription leads to an unambiguous value. Indeed, let C be expressed as a disjoint union of intervals in two ways, say, $\cup_n I_n$ and $\cup_m J_m$. Since intersection of two intervals is again an interval, using the countable additivity mentioned earlier, you can observe the following: for each n, $L(I_n) = \sum_m L(I_n \cap J_m)$ while for each m, $L(J_m) = \sum_n L(I_n \cap J_m)$. Thus both $\sum_n L(I_n)$ and $\sum_m L(J_m)$ equal $\sum_{m,n} L(I_n \cap J_m)$.

It is worth observing the following. If you have a sequence of intervals $\{I_n\}$, disjoint or not, their union is in the class $\mathcal{C}$. To see this, first observe that if $I_1 \cap I_m$ is non-empty, then $I_1 \cup I_m$ is an interval. More generally, you can say that I_m is linked to I_1, if you can get $n_1, \cdots, n_k$ such that any two consecutive intervals in the list $I_1, I_{n_1}, \cdots, I_{n_k}, I_m$ have non-empty intersection. If you consider all intervals in the given sequence which are linked to I_1 then their union is again an interval and the remaining intervals are disjoint with this union. You can now repeat this argument with one of the remaining intervals. The final upshot is a sequence of disjoint intervals having the same union as the given sequence.

In passing, let us mention that usually one considers intervals of the form $(a, b]$ with certain generalizations in mind. However we have no such compulsion. On the other hand, considering a particular type of interval may lead you to think that they have some special feature, where there is none.

2. Borel and Lebesgue

A natural question is whether this assignment of length can be made to more sets. Why should one be interested in such a question?

The French mathematician Emile Borel, in 1898, was considering functions $f(x) = \sum f_n(x)$, defined by a series. The specific series he considered, though important, is not our main point. He was interested to know how 'large' is the set of numbers x for which the series is convergent. Here large refers to its length. But his set is not in the class $\mathcal{C}$.

Here is another problem discussed by Borel. Consider the set of those numbers x in the interval $[0, 1]$ in whose decimal expansion each of the decimal digits occurs with frequency $1/10$. What is its length? He formulated the

It appears possible and useful to define length for more sets.

problem precisely and showed that the length of this set is one. This set is again not in the class $\mathcal{C}$.

Thus it appears possible and useful to define length for more sets than just those in the class $\mathcal{C}$. Let us accept that sets in the class $\mathcal{C}$ should have lengths as described above. Does this prescription unambiguously determine lengths of other sets? Yes, indeed a large class of sets have their lengths uniquely cut out. Let us see why.

Take $A \subset [0,1]$. Suppose $C \in \mathcal{C}$ and $A \subset C$, then clearly we should have $L(A) \leq L(C)$. Keep in mind that this inequality holds whenever we have C as above. Let b be the infimum of all such $L(C)$ as C varies over sets in $\mathcal{C}$ that contain A. We cannot assign any value larger than b as length of A. Thus, the maximum possible value for $L(A)$ is b. Similarly, suppose $A^c = [0,1] - A \subset D \in \mathcal{C}$. Then $L(A^c) \leq L(D)$. Let us denote by a the infimum of all such $L(D)$ as D varies over sets in $\mathcal{C}$ that contain A^c. Thus the maximum possible value for $L(A^c)$ is a. If we somehow define lengths of A and $[0,1] - A$, then it is sensible to expect that these two lengths add to one. In particular, the statement 'maximum possible value for $L(A^c)$ equals a' translates to 'minimum possible value for $L(A)$ is $1 - a$'. In other words, $1 - a \leq L(A) \leq b$.

Now comes a real beauty, an epoch-making idea of the French mathematician Henri Lebesgue in 1901. What if it so happens that $b = 1 - a$. In other words, the maximum possible value for $L(A)$ coincides with the minimum possible value. In such a case, there is no choice for us to make. If we want to define length of A then it has to be b, no more and no less. Generally, any prescription which arises naturally, without our intervention, must behave perfectly, right? We now proceed to explain that this is indeed the case. But let us hasten to add that we are not giving a historical survey; in particular, we take the liberty of not alluding to the attempts of Giusseppe Peano in 1883, Camille Jordan in 1893.

Let us return to the idea of Lebesgue. Collect all sets A for which assignment of length is not left to our choice, but is uniquely determined as above. In other words, the minimum value that we can assign coincides with the maximum value. Denote this collection of sets by $\mathcal{L}$. Here are some deep and profound conclusions of Lebesgue.

Firstly, this class of sets $\mathcal{L}$ satsfies three conditions:

(i) Every set that belongs to the class $\mathcal{C}$ also belongs to the class of sets $\mathcal{L}$. (ii) If a set A is in this class $\mathcal{L}$ then so is its complement $A^c = [0,1] - A$. (iii) If a sequence of sets (disjoint or not) is in this class then so is their union.

Secondly, let L be the length defined for sets in this class, namely, for $A \in \mathcal{L}$ its length $L(A)$ is the common value described above. Then this definition of L agrees, on the collection of intervals, with the notion of length we started with. Further L is countably additive, that is, if you take a sequence $\{A_n\}$ of disjoint sets in this class then $L(\cup A_n) = \sum L(A_n)$. Note that condition (iii) above already implies that the union $\cup A_n$ is in this class. In particular, if $C \in \mathcal{C}$ then the present value is the same as the value prescribed earlier.

This class of sets $\mathcal{L}$ has two further pleasing properties. Firstly, if a set $A \in \mathcal{L}$ has length zero, then every subset of A is also in $\mathcal{L}$ and has length zero, as it should be. Secondly, if a set $A \in \mathcal{L}$, then all its translates are also in $\mathcal{L}$ and receive the same length. This means the following. For $a \in R$, if we define $A + a = \{x + a \text{ (modulo 1)} : x \in A\}$, then $A + a \in \mathcal{L}$ and $L(A + a) = L(A)$. This is called the translation invariance property of length. Of course, for intervals this is obvious.

Lebesgue used the above observations to develop an integral, more general than that of Riemann's. It is worth

Length is translation invariant.

"It is to solve these problems, and not for the love of complications, that I have introduced ... ."
Lebesgue

recalling the words of the architect from *Introduction to 'Lecons sur L' intégration'* (taken from [5]):

One might ask if there is sufficient interest to occupy oneself with such complications, and if it is not better to restrict oneself to the study of functions that necessitate only simple definitions. ⋯ . As we shall see, in this course, we would then have to renounce the possibility of resolving many problems posed long ago, and which have simple statements. It is to solve these problems, and not for the love of complications, that I have introduced in this book a definition of the integral more general than that of Riemann.

3. Vitali and Gödel

The important question now is whether every subset of $[0, 1]$ is in the class $\mathcal{L}$. The Italian mathematician Giuseppe Vitali used the Axiom of Choice (AC), in 1905, to construct a set which is not in this class. AC has many equivalent formulations and here is one. If you have a (non-empty) collection of non-empty disjoint sets, you can make another set which has exactly one element in common with each set in the given collection.

You might say, why does one need an axiom for this, just pick any one element from each set and make a new set. The trouble is that in the absence of a rule it is not clear which element should be chosen. This point is important because, unless we know which elements are in our set, and which are not, how can we say we have got a set. It is worth recalling an example of Bertrand Russell. We have an infinite collection of pairs of new shoes and want to choose one shoe from each pair. We have an infinite collection of pairs of new socks and want to choose one from each pair. You can prescribe an algorithm for the first but not for the second. Think about it.

Returning to Vitali's construction, decompose the interval $[0, 1]$ into disjoint subsets, each countable, as follows.

Say $x \sim y$ if $x - y$ is rational, and consider the equivalence classes. Each of these sets is countable; it actually consists of a number along with all its (modulo one) rational translates. Select one element from each equivalence class, denote the resulting set by S. This set is not in the collection $\mathcal{L}$. Let us see why.

The key observation is the translation invariance property mentioned earlier. Suppose $S \in \mathcal{L}$. Can it have length zero? No, because all its countably many translates get length zero and they all make up $[0, 1]$. Can it have length strictly positive? No, because then all its infinitely many disjoint translates have the same length whereas length of $[0, 1]$ is one. See how countably additivity of length is crucial for this argument.

In the above construction, we have not been able to prescribe any algorithm for selecting a point from each equivalence class and we needed the axiom of choice. Are there better axioms that allow us to *describe* a set that is not in $\mathcal{L}$?

The Austrian logician Kurt Gödel[1] introduced, in 1938, a set theoretic axiom called the axiom of constructibility – another epoch-making discovery. By set theory we mean the usual Zermelo–Fraenkel theory, known as ZF. However, it is not necessary for you to know what it is. It is just what you and I live in. He showed that his new axiom is consistent with the other axioms. This simply means the following. He gave an algorithm so that any argument that uses this new axiom and leads to a contradiction can be translated into an argument which does not use this new axiom and yet leads to a contradiction. AC is a consequence of this new axiom and hence AC is also consistent. This new axiom allows, as shown by Gödel in 1940, to specifically describe one set which is not in $\mathcal{L}$.

[1] See *Resonance,* Vol.6, No.7, 2001.

Any argument that uses this new axiom and leads to a contradiction can be translated to an argument which does not use this new axiom.

For the uninitiated, let us add the following. Describing a set A means making a statement about numbers, using

Can we associate length for a larger class?

only the usual symbols, such that the set A consists of precisely those numbers for which the statement is true. For those initiated in logic, describing A means being able to write a formula with one free variable x such that A consists of precisely those numbers that satisfy the formula.

4. Ulam

The next question is the following. Can we assign length for a larger class of sets $\mathcal{M}$ satisfying the conditions (i), (ii), (iii) of $\mathcal{L}$ and so that it is countably additive? The answer is yes. Take any set $S \notin \mathcal{L}$. By definition of the class $\mathcal{L}$, we conclude $a(S) < b(S)$. Here $a(S)$ is the minimum value and $b(S)$ is the maximum value possible for length of S. Enlarge $\mathcal{L}$ to the smallest class $\mathcal{M}$ of sets satisfying the three conditions (i), (ii), (iii) listed above and such that $S \in \mathcal{M}$. A careful hands-on calculation shows that we can extend length to this class by assigning *any* value between $a(S)$ and $b(S)$ as the length of S. In particular, such extension is not unique.

One intriguing question is: can you associate length for *every* subset of I? In other words, can you associate with every subset $A \subset [0,1]$, a number $L(A)$, in such a way that (a) $L(\cup A_n) = \sum L(A_n)$ for pairwise disjoint sequence of sets (A_n) and (b) if A is an interval, then $L(A)$ equals length of the interval A. We have used the same symbol L, as used earlier, because of the following reason. The two requirements force that any such notion of length coincide with the value above for all sets in $\mathcal{L}$; just look at the definition of the class $\mathcal{L}$. The Polish mathematician Stanislaw Ulam showed, in 1930, that (under reasonable axioms) such an assignment of length for all subsets is not possible. To explain this we start by recalling a well-known property of the set N of natural numbers.

The usual relation '$\leq$' between pairs of natural numbers satisfies the following properties: (1) $x \leq x$ for all x; (2) $x \leq y$ and $y \leq x$ implies $x = y$; (3) $x \leq y$ and $y \leq z$ implies $x \leq z$ and (4) $x \leq y$ or $y \leq x$ for all x, y. The infinite system $(N, \leq)$ has the following additional property.

(5^*) For every $x \in N$, the set $\{y : y \leq x\}$ is finite.

Question: Is it possible to prescribe such an order relation $\leq$ between pairs of points of $I = [0, 1]$? Clearly, it is not possible because, (5^*) implies that the set I must be countable. This can be seen as follows.

Suppose you can define such a relation. Define for any $x \in I$, the initial segment $S(x) = \{y : y \leq x\}$. Remember this is a finite set by condition (5^*). Start with an element $x_1 \in I$. If $S(x_1)$ is not all of I, pick x_2 which is not in $S(x_1)$. Observe that $S(x_2) \supset S(x_1)$. If $S(x_2)$ is not all of I, pick x_3 not in $S(x_2)$. If you stop at a finite stage, you clearly see that I is finite. If you continue for all $n \geq 1$, put $A = \cup S(x_n)$. If this is all of I, you see that I is countable. If this is not all of I pick a point x not in A to see that condition (5^*) is violated for this point x, simply because $A \subset S(x)$.

So, to make a meaningful requirement we modify the last property to the following.

(5) For any $x \in I$, the set $\{y : y \leq x\}$ is countable (finite or infinite).

The usual order relation among real numbers does not satisfy condition (5). Let us make a hypothesis.

(H1) There is an order relation defined for pairs of points of I satisfying the five conditions (1–5).

Suppose that **(H1)** holds. We shall exhibit a matrix $(A_{x,n})$. The entries of the matrix will be subsets of I. Thus each $A_{x,n}$ is a subset of I. It will have countably infinitely many columns. In fact, n runs over $\{1, 2 \cdots\}$. It will have uncountably many rows. In fact, x runs over I itself. This matrix of sets will have the following two

We shall exhibit a matrix of sets.

properties.

(α) Sets in any given column are pairwise disjoint.

$$A_{x,n} \cap A_{y,n} = \emptyset; \quad \text{for} \quad x \neq y;\ n \geq 1\ .$$

(β) Sets in any given row make up I except for a countable set.

$$\bigcup_n A_{x,n} = I - \{\text{a countable set}\} \quad \text{for} \quad x \in I.$$

What is the use of such a matrix of sets? Suppose L can be defined for all subsets satisfying the stated requirements. Mark a set red if it has non-zero length. Remember that countable sets have length zero. Property (β) says that there is at least one red set in each row. Since there are uncountably many rows, there must be uncountably many red sets. But there are only countably many columns. So one column must contain uncountably many red sets. But then property (α) says that there are uncountably many disjoint sets of non-zero length. This is impossible. Just note that for each integer $k \geq 1$, there cannot be more than k disjoint sets, whose length exceeds $1/k$.

In passing, let us note that in the argument above we did not use any deep property of length. We only needed that singleton sets have length zero; length should be countably additive; length of $[0, 1]$ is one. Thus we cannot assign a number for each subset of $[0, 1]$ so that these three conditions are satisfied.

Here is how to construct such a matrix of sets. For each $y \in I$, let φ_y be a one-one function from the countable set $\{z : z \leq y\}$ into/onto $\{1, 2, \cdots\}$. Since we do not know whether there will be infinitely many points smaller than y (in this order) we have to allow the map to be into. Also keep in mind that the function φ_y is not defined on all of I. For $x \in I$ and $n \geq 1$, define the set, $A(x, n) = \{y : \varphi_y(x) \text{ is defined and equals } n\}$.

If $y \in A(x,n) \cap A(x',n)$, then $\varphi_y(x) = \varphi_y(x') = n$ and φ_y being one-to-one map we conclude that $x = x'$. Finally, fix x, and consider $\bigcup_n A(x,n)$. If $x \leq y$ then φ_y is defined at x and if $\varphi_y(x) = n$, then $y \in A(x,n)$. Thus the union under consideration includes all points y except those in the countable set $\{y : y \leq x; y \neq x\}$. We can take $A(x,n)$ as $A_{x,n}$.

Without any further axiom, it is not possible to define length for all subsets.

The question that arises is whether hypothesis **(H1)** is reasonable. Yes, it is. If you assume the Axiom of Choice (AC) and the Continuum Hypothesis (CH) then **(H1)** follows. The axiom of constructibility of Gödel mentioned earlier yields both of these as consequences. AC was already mentioned earlier. CH is the statement that if we take any subset of I, then either it is a countable set or it has the same number of points as I. This last statement just means that there is a one-one function defined on the given set onto I.

Thus without any further axioms, it is not possible to define length for all subsets. At this stage you might ask, can you name one set for which we cannot define length. NO, we cannot. Given any set, as mentioned at the beginning of this section, we can define length for that set. It is similar (but more subtle because you are allowed to use fractions) to the following. You have 15 apples and there are 16 students in the class. You announce that you cannot provide one apple to each student. True. However, if any one student is named, you can definitely give an apple to her.

5. Kakutani, Kodaira and Oxtoby

The next natural question is how far can it be extended? Initial results in this direction were due to the Japanese mathematicians Shizuo Kakutani and Kunihiko Kodaira during 1944–50. Here is the final conclusion of Kakutani and the American mathematician John Oxtoby, in 1950. We can get a 'huge' collection of sets $\mathcal{M}$ larger than $\mathcal{L}$

We can associate translation-invariant length for a huge class of sets.

satisfying the conditions (i), (ii), (iii) of $\mathcal{L}$ along with the additional property: whenever $A \in \mathcal{M}$, so is any of its translates. We can associate length L for all sets in this class such that it is countably additive and also translation invariant. We shall now explain what is meant by 'huge'. After you understand this, you will realize that this is the best possible. This construction is involved and uses AC and the best source is the original paper.

There are uncountably many points in the interval $[0, 1]$. However, given any number, we can get a rational number as close to it as we want. More precisely, if $x \in [0, 1]$ and $\epsilon > 0$ then there is a rational number r such that $d(x, r) < \epsilon$. Here $d(x, y) = |x - y|$ denotes the distance between the points x and y. One says that the countable set Q of rational numbers is a dense set.

A similar situation obtains with the class $\mathcal{L}$. There is a countable family of sets $\mathcal{L}_0 \subset \mathcal{L}$ with the following property: given any $A \in \mathcal{L}$ and $\epsilon > 0$ there is $B \in \mathcal{L}_0$ such that $L(A\Delta B) < \epsilon$. Here the notion of distance is $d(A, B) = L(A\Delta B)$. Recall $A\Delta B$ is the set of points that belong to exactly one of these sets; in symbols it is $(A - B) \cup (B - A)$ or equivalently, $(A \cup B) - (A \cap B)$. It is easy to verify that (i) $d(A, A) = 0$; (iii) $d(A, B) = d(B, A)$ and $d(A, C) \leq d(A, B) + d(B, C)$. Of course, $d(A, B) = 0$ does not imply that the two sets A and B are the same. You can observe this by taking any two singleton sets.

Returning to the collection $\mathcal{M}$ exhibited by Kakutani and Oxtoby, we want a 'dense' subcollection $\mathcal{M}_0 \subset \mathcal{M}$: given $A \in \mathcal{M}$ and $\epsilon > 0$ there is $B \in \mathcal{M}_0$ with $L(A\Delta B) < \epsilon$. Of course you can take $\mathcal{M}_0 = \mathcal{M}$. But we want to know how small such a set $\mathcal{M}_0$ can be. The answer is: the family $\mathcal{M}_0$ should be as large as the collection of all subsets of $[0, 1]$.

To use symbols, recall that $\aleph_0$ denotes the number of points in N, the set of natural numbers. The set $[0, 1]$ is not countable and the number of elements in this set is denoted by **c**. If $\mathcal{P}$ denotes the collection of all subsets of $[0, 1]$, then it is uncountable, but has more points than $[0, 1]$. This means the following. You can define a one-to-one map on $[0, 1]$ into $\mathcal{P}$, but you cannot define a one-to-one map of $[0, 1]$ *onto* $\mathcal{P}$. The number of points in $\mathcal{P}$ is denoted by $\mathbf{2^c}$. Here then is the property of the family $\mathcal{M}$ of Kakutani–Oxtoby: any dense subset must have $\mathbf{2^c}$ many elements. This is the meaning of saying that $\mathcal{M}$ is huge.

6. Steinhaus, Mycielski and Swierzkowski

Even though we can assign length to a huge class of sets, such an assignment lacks any uniqueness property. So it is always interesting to see if every subset is in the class $\mathcal{L}$ itself. Of course, the results mentioned in Section 3 imply that if we assume certain axioms, then not every set is in $\mathcal{L}$. Are there other axioms under which every subset of $[0, 1]$ is in $\mathcal{L}$?

The Polish mathematicians Hugo Steinhaus and Jan Mycielski introduced, in 1961, an axiom called the Axiom of Determinatenes (AD). It says that in certain two player games there is always a winning strategy to one of the players. Here is a precise description. You and I successively and alternately choose positive integers, me choosing first. This is a game of perfect information, in the sense that at every stage we both know all the choices made so far before the next move. Thus we produce an infinite sequence of points of N, that is, an element of the product space N^N. Suppose a set $A \subset N^N$ is prescribed in advance, before the game starts. If the resulting sequence lies in this set A then I win, otherwise you win.

A strategy for you is a rule that dictates how you should play at each stage. More precisely, it is a function τ which associates a positive integer with every odd-length finite sequence of integers. The idea is the following. If

Strategy for you is a rule that dictates how you should play.

Assuming a reasonable axiom, Solovay showed consistency.

the present position of the game is the finite sequence s of odd length, then it is your turn to play and you choose $\tau(s)$. The strategy τ is a winning strategy for you if, no matter how I make my choices, you win by using τ. Similarly we can define strategy and winning strategy for me. Keep in mind that a strategy for me should be defined on sequences of even-length, including the empty sequence (this tells me how to start). Here then is the axiom AD: Given any subset $A \subset N^N$, either you have a winning strategy or I have a winning strategy. It is easy to see that, in any given game, both of us cannot have winning strategies.

Assuming AD, Mycielski and Swierzkowski showed, in 1964, that every subset of $[0, 1]$ is in the class $\mathcal{L}$. What is the status of this axiom? We cannot prove its consistency in ZF.

7. Solovay

Consider the statement **S**: 'every subset of $[0, 1]$ is in the class $\mathcal{L}$'. We saw that the axiom AD implies **S**. This is unsatisfactory because we cannot prove consistency of AD. Can we *directly* show consistency of the statement **S**? Unfortunately this cannot be done, as shown by the American logician Robert Solovay in 1967. He did the next best thing in 1964. Assuming a reasonable axiom, he showed the consistency of **S**. Statements of these results take us into deeper realms of logic and set theory.

Suggested Reading

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