Minimum Linear Velocity and Maximum Angular Velocity in a Pursuit Problem: Use of Calculus

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A ship/boat is travelling parallel to a river bank and a patrol vehicle is approaching the river along a road perpendicular to the river bank, i.e., the ship and the vehicle are moving at right angles to each other with uniform speeds. The patrol vehicle focuses a search light on the ship at all time instants, implying continuous rotation of the search light beam. In this article the relative velocity and acceleration between the ship and the vehicle as well as the angular velocity and acceleration of the beam are determined. Some numerical examples are also cited.

1. Introduction

The angular velocity of a search light beam focused on a person moving along a straight path, by a patrol positioned at a fixed distance from that straight path has been determined [1].

Here we consider a harder and improved problem wherein both carrier of search light and focused object are moving perpendicular to each other with uniform speeds. From a realistic point of view let us suppose that a ship is sailing with a uniform speed v along a straight path parallel to the river bank and a patrol car is moving with a uniform speed u along a road at right angles to the bank while casting a search light on the ship. The line of sight of the ship from the vehicle (i.e., the search light beam) rotates with variable angular velocity with respect to the latter's path. However, the entire situation can be depicted in the following way.

Consider a system of axes XOY (OX $\perp$ OY), where O is the origin. A vehicle is moving with a uniform speed

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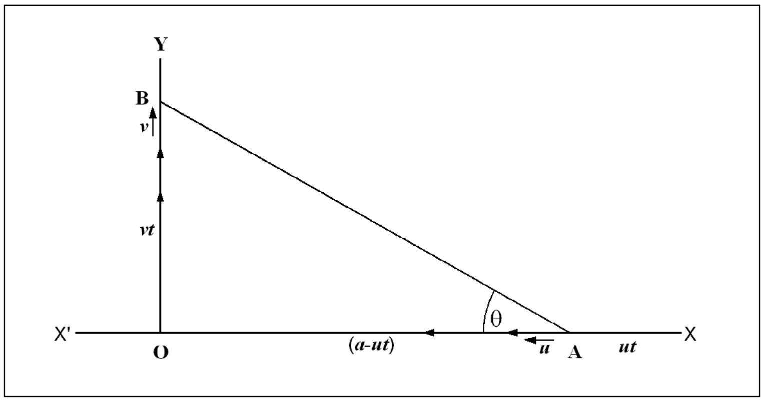
Figure 1.



u along the x-axis (OX) and a ship is moving with a speed v along the y-axis (OY). When the vehicle is at a distance a from the origin O, it sights the ship at the origin and begins to cast a search light on the ship and holds the beam along the line of sight of the ship at all time instants. In order to do this, the beam is rotated with a variable angular velocity with respect to the vehicle path, i.e., x-axis. This is illustrated in *Figure* 1.

2. Angular Velocity and Deceleration Along with their Maximum Values of the Light Beam

If s is the distance between the patrol vehicle and the ship, i.e., length of the light beam, and θ is the angle made by the light beam with the vehicle path at any instant of time t, then

$$s^2 = (a - ut)^2 + (vt)^2, \qquad (1)$$

$$\tan\theta = \frac{vt}{a - ut} \ . \qquad (2)$$

The beam is rotated with a variable angular velocity with respect to the vehicle path

Differentiating (2) with respect to time t, we get the angular velocity ω of the light beam at any time t:

$$\sec^2\theta \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{(a - ut)v + uvt}{(a - ut)^2}, \qquad (3)$$

or,

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{av}{(1+\tan^2\theta)(a-ut)^2}$$

by use of (2) and (1); that is,

$$\omega = \frac{av}{s^2}. \tag{4}$$

This suggests that the angular velocity of the light beam is inversely proportional to the (distance)^2 between the vehicle and the ship, i.e., (length of the beam)^2. So the angular velocity is maximum when this distance is the least.

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Many textbooks on mechanics [1,2] find the least distance between two moving objects but not angular velocity of the line of sight, which has been tackled here using calculus, geometry and trigonometry. However, here we find the minimum value of s by completion of square from (1):

$$\begin{aligned} s^2 &= a^2 + (u^2+v^2)t^2 - 2aut = a^2 + (u^2+v^2) \\ &\quad \times \left[t^2 - \frac{2aut}{u^2+v^2} + \frac{u^2a^2}{(u^2+v^2)^2}\right] - \frac{u^2a^2}{u^2+v^2} \\ &= a^2 + (u^2+v^2)\left(t - \frac{ua}{u^2+v^2}\right)^2 - \frac{u^2a^2}{u^2+v^2}. \end{aligned} \tag{5}$$

Hence the minimum value of s occurs when

$$t_{\mathrm{opt}} = \frac{ua}{u^2+v^2}\ ; \tag{6}$$

$$s_{\min} = \frac{av}{\sqrt{u^2+v^2}}\ . \tag{7}$$

Using (4) one gets the maximum angular velocity of the beam, i.e., line of sight of the ship from the vehicle:

$$\omega_{\max} = \frac{u^2+v^2}{av} \tag{8}$$

at the time given by (6).

Relative acceleration is inversely proportional to the cube of the distance between them and acquires a maximum value when this distance is minimum

Differentiating (5) with respect to time t, one gets the relative velocity between the patrol vehicle and the ship or in other words the rate of lengthening of the light beam or the line of sight:

$$v_{\mathrm{R}} = \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{(u^2+v^2)}{s}\left(t - \frac{ua}{u^2+v^2}\right) \quad \text{(by use of (5))}$$

$$v_{\mathrm{R}} = \frac{1}{s}\sqrt{\left(s^2 - a^2 + \frac{u^2a^2}{u^2+v^2}\right)(u^2+v^2)} = \sqrt{u^2+v^2-\frac{v^2a^2}{s^2}} \tag{9}$$

which can be expressed in terms of angular velocity because of (4)

$$v_{\mathrm{R}} = \sqrt{u^2+v^2-a\omega v}. \tag{10}$$

By use of (9) and (4) one can find the relative acceleration

$$f_{\mathrm{R}} = \frac{\mathrm{d}^2s}{\mathrm{d}t^2} = v_{\mathrm{R}}\frac{\mathrm{d}v_{\mathrm{R}}}{\mathrm{d}s} = \frac{1}{2}\frac{\mathrm{d}v_{\mathrm{R}}^2}{\mathrm{d}s} = \frac{v^2a^2}{s^3}. \tag{11}$$

Interestingly this relative acceleration is inversely proportional to the cube of the distance between them and acquires a maximum value when this distance is minimum, owing to (7):

$$(f_{\mathrm{R}})_{\max} = \frac{(u^2+v^2)^{3/2}}{av}. \tag{12}$$

If ϵ is the rate of change of angular velocity of the light beam, i.e., the line of sight at unit time, then in consequence of (4), one gets

$$\epsilon = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{-2av}{s^3}\frac{\mathrm{d}s}{\mathrm{d}t} \quad \text{(by use of (9))}$$

or

$$\epsilon = \frac{-2av}{s^3}\sqrt{(u^2+v^2)-\frac{a^2v^2}{s^2}} \tag{13}$$

which represents angular acceleration or deceleration according as v_{R} given by (9) is < 0 or > 0, and is inversely proportional to the cube of the distance s of separation between the two moving objects. This vanishes when the distance of separation is minimum as is evident from (7). Nevertheless, it can be established that this can attain a maximum value; so (13) is rewritten as

$$|\epsilon| = 2av\left(\frac{u^2+v^2}{s^6} - \frac{a^2v^2}{s^8}\right)^{1/2}. \tag{14}$$

For maximum or minimum of ϵ,

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}s} = 0\,.$$

Therefore $av\left[\dfrac{-6(u^2+v^2)}{s^7} + \dfrac{8a^2v^2}{s^9}\right] = 0$

$$\text{which gives} \quad s^2_{\text{opt}} = \frac{4a^2v^2}{3(u^2+v^2)}. \tag{15}$$

$$\begin{aligned}
\text{Now,}\quad \frac{\mathrm{d}|\epsilon|}{\mathrm{d}s} &= \frac{2a^2v^2}{|\epsilon|}\left[\frac{-6(a^2+v^2)}{s^7} + \frac{8a^2v^2}{s^9}\right],\\
\text{and}\quad \frac{\mathrm{d}^2|\epsilon|}{\mathrm{d}s^2} &= \frac{2a^2v^2}{|\epsilon|}\left[\frac{42(u^2+v^2)}{s^8} - \frac{72a^2v^2}{s^{10}}\right.\\
&\quad \left. + \left(\frac{8a^2v^2}{s^9} - \frac{6(u^2+v^2)}{s^7}\right)\frac{1}{|\epsilon|}\frac{\mathrm{d}|\epsilon|}{\mathrm{d}s}\right]\\
&= \frac{2a^2v^2}{|\epsilon|s^{10}}[42s^2(u^2+v^2) - 72a^2v^2]\\
&= -\frac{16\times 2a^4v^4}{|\epsilon|s^{10}} < 0\,.
\end{aligned}$$

Hence this maximum deceleration occurs when the optimum distance s_{opt} of separation is given by (15), and is given by

$$\begin{aligned}
|\epsilon_{\max}| &= \frac{2av}{s^4_{\text{opt}}}[s^2_{\text{opt}}(u^2+v^2) - a^2v^2]^{1/2}\\
&= \frac{2av}{s^4_{\text{opt}}}\left(\frac{4a^2v^2}{3} - a^2v^2\right)^{1/2} \quad \text{or}\\
|\epsilon_{\max}| &= \frac{3\sqrt{3}(u^2+v^2)^2}{8a^2v^2}\,,
\end{aligned} \tag{16}$$

where (15) has been used. Now the time at which this maximum angular deceleration arises is calculated using (15) and (5).

$$t_{\text{opt}} = \frac{a}{(u^2+v^2)}\left(\frac{v}{\sqrt{3}} + u\right). \tag{17}$$

The relative velocity, i.e., rate of increase in the search light length is obtained by use of (14) in (9) when it attains the maximum angular deceleration:

$$v'_{\text{R}} = \sqrt{u^2+v^2}/2. \tag{18}$$

3. Discussion

We analyze the situation when both vehicle and ship move with the same speed. Then $u = v$, which in consequence of equations (4) to (9) yields

$$s^2 = a^2 + 2ut(ut-a), \quad s_{\min} = \frac{a}{\sqrt{2}}, \quad \omega_{\max} = \frac{2u}{a}. \tag{19}$$

Further it is observed from (8) that the maximum angular velocity increases with increase in uniform velocity of the patrol vehicle, but acquires a minimum value if the speeds of the patrol vehicle and of the ship are equal in that

$$\frac{\mathrm{d}}{\mathrm{d}v}\omega_{\max} = 0 \Rightarrow$$
$$u = v \quad \text{along with } \frac{\mathrm{d}}{\mathrm{d}v^2}(\omega_{\max}) > 0.$$

However, the foregoing analysis is feasible and consistent if in the pursuit problem, one object is proceeding towards the junction (intersection of the travelled paths) and the other object is moving away from the junction. This is evident from *Figure* 1. Equations (1) to (4) indicate that the initial angular velocity $= \omega_0 = \frac{v}{a}$.

When the vehicle reaches the bank, the ship has travelled a distance equal to av/u and the angular velocity of the light becomes equal to $u^2/(va)$, because of (4).

Since $ua/u^2 + v^2 < a/u$, i.e., from (6) the time of minimum distance between them is less than the time of reaching the bank by the vehicle, this minimum distance occurs before the patrol vehicle reaches the bank. This minimum distance and the maximum angular velocity occur simultaneously.

Minimum distance and the maximum angular velocity occur simultaneously.

4. Numerical Example

Suppose u = velocity of the patrol vehicle = 120 km/hr, v = velicity of the ship = 50 km/hr, a = initial distance between the vehicle and the ship = 13 km. Then by use of the above formulation, we get as follows:

The minimum distance between the vehicle and the ship $= s_{\min} = 5$ km and the corresponding time $= t_{\text{opt}} = 5$ minutes 32 seconds; at this time the light beam attains the maximum angular velocity $= \omega_{\max} = 26$ rad/hr and also has the maximum relative acceleration $(f_{\text{R}})_{\max} = 3380$ km/hr^2. The beam gets the maximum angular deceleration after time $\left(\frac{72\sqrt{3}+30}{13\sqrt{3}}\right)$ minutes and this maximum $= \epsilon_{\max} = (253.5)\sqrt{3}$ rad/hr^2.

For two objects A and B moving at right angles to each other, as shown in *Figure* 1 with speeds u and v respectively, let A cross the junction O when B is at a distance b from O. Then the distance s between them and their relative velocity v_{R} at any time t reckoned from the instant of crossing O is:

$$s = \sqrt{(b+vt)^2 + u^2t^2} \quad \text{and} \quad v_{\text{R}} = \sqrt{u^2 + v^2 - \frac{b^2u^2}{s^2}}$$

and hence further analysis in line with the foregoing one can be carried out.

Suggested Reading

[1] David R Duncan and Bonnie H Litwiller, Related Time Rates: Comparing Angular and Linear Velocity, *Mathematics Teacher*, Vol.43, No.3 and 4, pp.193–195, 2007.

[2] M Ray and G C Sharma *A Textbook on Dynamics*, S Chand and Company Ltd., New Delhi 110055, pp. 21–24, 2000.