Multiplication: From Thales to Lie[1]

P H Keskar

The ancient Greek mathematician Thales of Miletus devised a method to make large measurements like the heights of pyramids. The straightedge or ruler construction of multiplication can be traced back to this method of Thales. The ruler construction also exists for addition. In this article, we indicate how the comparison between these two constructions leads to Lie theory, a device created by Lie to reduce the multiplicative problems to the additive ones. It is hoped that this enhances the interest in these two themes.

Pradipkumar H Keskar did his PhD in mathematics from Purdue University, USA. He has worked in Mehta Research Institute (now called Harish Chandra Research Institute) Allahabad, University of Mumbai and University of Pune. He is currently an Associate Professor in Birla Institute of Technology and Science, Pilani. His research interests are algebraic geometry, and Galois theory.

[1] This article is based on a talk presented at a conference in Pune commemorating Professor Abhyankar's 80th birthday.

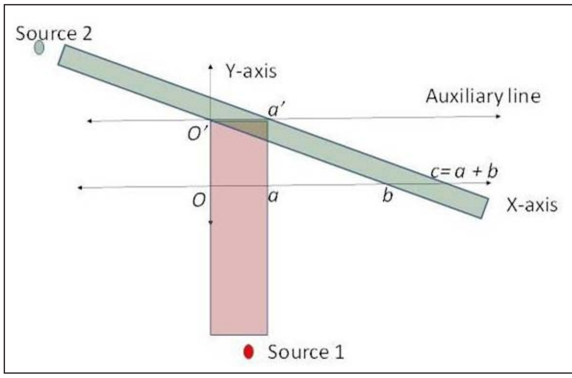
Keywords
Ruler constructions, addition, multiplication, unslanting, infinitesimals, Lie algebra, exponential map.

1. Geometric Constructions for Addition and Multiplication

Addition

To describe the geometric constructions of addition, as is expected, we represent the numbers by the points of a line in one-to-one manner, say the points of the X-axis in the (X, Y)-plane. A specific point on the line is fixed, to correspond with 0, the additive identity, and labeled as O.

The geometric construction to get the sum $a+b$ of numbers a, b is depicted in *Figure* 1, which we describe as follows: Draw an auxiliary line parallel to the X-axis. Place a source of light, Source 1, at an infinite distance in any direction other than the direction of the X-axis. The rays from Source 1 will be parallel to each other. Take the shadow of the segment Oa on the auxiliary line. Let O', a' be respectively the shadows of O, a on the auxiliary line. Now place another source, Source 2, of light at infinite distance in such a direction that the shadow of O' on the X-axis is at b. Then the shadow of a' on the X-axis is the point $c = a + b$. This construction is based upon the principle that the opposite



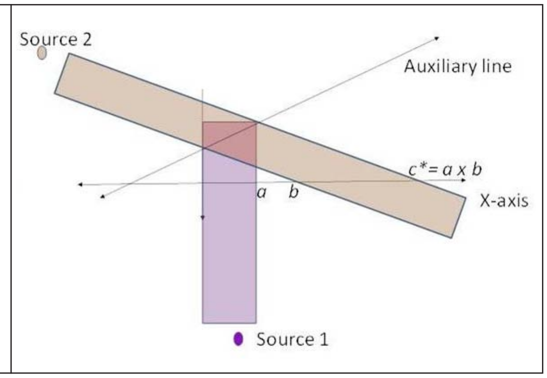


Figure 1 (left).

Figure 2 (right).

sides of a parallelogram have the same length. In effect, the shadow of a shadow transports the segment Oa on the X-axis so that O is taken to b and hence performs a rigid motion of translation. This explains why this construction gives $a+b$.

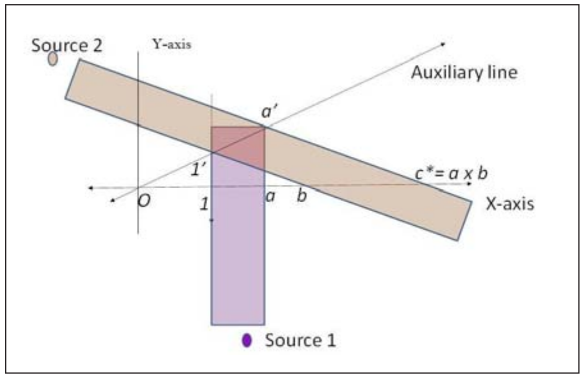
A Slant

What if the auxiliary line is chosen to be a line not parallel to the X-axis? Abracadabra! Magically, the answer is $c^* = a \times b$ rather than $a+b$. Thus a slant in auxiliary line produces a slant in '+' to make it '×'! This is illustrated in *Figure* 2[2].

[2] This magical act was used by me for the quick passage from geometric construction of addition to that of multiplication in a presentation given to high school students on the occasion of the Science Day (28-2-2008) in BITS, Pilani. The unraveling of this magic is the genesis of this article.

As with any magic, there is slight cheating here! This is hidden in the ambiguity resulting from the unlabeled points in *Figure* 2. For the geometric construction of multiplication, one requires to fix the multiplicative identity 1. To accommodate it, we need to relabel the points of the X-axis. In fact, instead of Oa, initially we consider the shadow of segment $1a$. Thus the original O becomes 1. The new O is chosen as the point of intersection of the auxiliary line with the X-axis. Using elementary high school geometry, one can explain why the resulting point c^* is $a \times b$. In *Figure* 3, which is the updated version of *Figure* 2, each of the pairs $(O11', Oaa')$ and $(Ob1', Oc^*a')$ is a pair of similar triangles – hence their corresponding sides are proportional. As a consequence of this fact, we get the desired conclusion that $c^* = a \times b$. This picture is also a key to Thales's method of measuring the heights of pyramids.

Figure 3..



For further details regarding the key role of these constructions in synthetic projective geometry, refer to the wonderful book on projective geometry by A Seidenberg [8].

The cheating though, is not even as grave as the 'Naro va Kunjaro va' cheating in *Mahabharat*[3]. Indeed, as explained above, if we slant the auxiliary line in *Figure* 1 but don't label the points, the resulting binary operation is a multiplication, but in some other coordinate system. In fact, depending on where the slanted auxiliary line meets the X-axis, we get several binary operations, i.e. a family of binary operations, these being multiplications in different coordinate systems.

[3] Here we are referring to the endorsement by Yudhishthir, the Pandava prince, of the rumor 'Ashwatthama is dead'. It was the death of an elephant, who shared his name with the son, a warrior himself, of the Kauravas' general. This turned the war around leading to Yudhishthir's victory but, as a consequence of this cheating, the chariot of the virtuous Yudhishthir, which used to float some inches above the ground, touched the ground. Was it the slant in auxiliary line which resulted in this? May we assume that Thales's method was inspired by his reading of *Mahabharat* presuming *Mahabharat* was written before his era?

Multiplication and the Coordinate Change, Use of an Algebraic Identity

Consider the coordinate change $T(x) = -t(x-1)$. Then $T(0) = t,\ T(1) = 0$. Let $u = T(a),\ v = T(b)$. In terms of the new coordinates, the binary operation of multiplication transforms to a new operation $\otimes$ such that $u \otimes_t v = T(ab) = -t(ab-1) = -t\left[\left(1-\frac{u}{t}\right)\left(1-\frac{v}{t}\right)-1\right] = (u+v) - \frac{uv}{t}$, by using an*algebraic identity*

$$(1+x)(1+y) = 1 + x + y + xy.$$

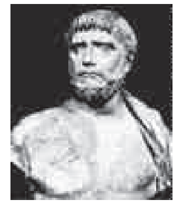
Unslanting, Projective Geometric View

Nonetheless, there is a striking similarity between the two geometric constructions of addition and multiplication. Let us extend the ambient space of the construction to a projective plane, by adding to it a projective line. In the projective plane any two lines meet, the parallel lines meet at the points of line at infinity, the newly added line. Hence even in the geometric construction of addition, the auxiliary line meets the X-axis, thus blurring the distinction between the addition and the multiplication. One may wonder if *Figure* 1 can be transformed to *Figure* 2 by applying a collineation of the projective plane?[4] Let us say that a binary operation on the points of X-axis in the projective plane is a version of the multiplication of the points of X-axis if it can be obtained from multiplication by relabeling the points of X-axis through a collineation of the projective plane which maps the (projective) X-axis to itself. Note that the coordinate change considered above in the definition of $u \otimes_t v$ comes from a collineation of the projective plane given by the fractional linear transformation $(x, y) \mapsto (-t(x-1), y)$, and hence $\otimes_t$ is a version of the multiplication of the points on X-axis. Is the addition also just another version of the multiplication?

A closer comparison of *Figure* 1 and *Figure* 2 brings out a subtle difference between the two pictures. In *Figure* 1 considered in projective plane, both the sources of light are collinear with the point of intersection of the X-axis with the auxiliary line, viz., they all lie on the line at infinity. On the other hand, in *Figure* 2, they are not collinear.

But then we can imagine that the situation of collinear sources of light is a limiting case of the situation of non-collinear sources of light. This amounts to the unslanting process, where a pair of intersecting lines in the affine case gets converted to a pair of parallel lines in the affine plane. In fact, we are considering the family $\otimes_t$ of binary operations as t varies and want to see what happens to the resulting operation in the limiting case as $t \to \infty$. To see this algebraically, in the expression for $u \otimes_t v$, as $t \to \infty$, the last term, with t in the denominator can be

Box 1. Thales of Miletus



This ancient Greek mathematician-cum-philosopher lived from around 624 BC to 546 BC. Born in the city of Miletus on west coast of Asia Minor, some legends have it that he traveled to Egypt and transported geometry from there to Greece. Coming from the era before Euclid and even Pythagoras, his understanding and contributions to geometry are commendable. Of particular relevance to us is his understanding of similar triangles, as is apparent from the method he developed to measure heights of pyramids.

[4] Recall that a collineation of a projective plane is a permutation of its points preserving the collinearity, as well as non-collinearity, of points. Its importance is derived from the fact that projective geometry studies those properties which can be expressed in terms of collinearity of point sets.

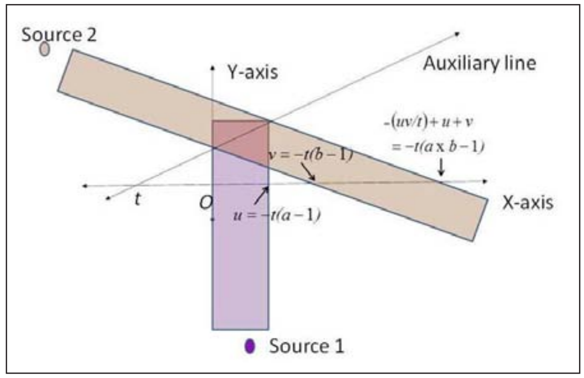


Figure 4.

ignored, viz., $\lim_{t\to\infty} u \otimes_t v = u + v$ is what we get! See *Figure* 4.

Wait a minute! The coordinates u and v depend on t and their limit must also be taken. Both approach ∞. So does $u \otimes_t v$. This is equivalent to the not so interesting identity $\infty + \infty = \infty$. We need a remedy.

2. Infinitesimals and Lie Algebra

Recovery by the Use of Infinìnitesimals

Things can be somewhat recovered, with the use of infinitesimals. The concept of infinitesimals has always been intriguing, making its history interesting. The first systematic use of infinitesimals was made by Newton in his theory of fluxions, the original name for derivatives. It soon came under criticism, the strongest critic being Bishop George Berkeley[5], which prompted Newton to refine the concept of infinitesimals.

Still the intrigue continued till Cauchy and Abel formulated the concept of limits (or convergence) in terms of ϵ and δ. This approach of Cauchy and Abel became more and more abstract, and surprisingly, tides turned back to Newtonian ideas due to the work of Krull and Chevalley on the I-adic topologies on rings with an ideal I, resulting in the strengthening of the original approach to the infinitesimals[6]. Later, the concept of infinitesimals was still further generalized by Grothendieck in arbitrary schemes by considering schemes having nonzero

[5] The following quote of Bishop Berkeley (see [7], p. 326) indicates the nature of his critique: "And what are these fluxions? The velocities of evanescent increments. And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them *ghosts of departed quantities*?"

[6] To quote from Abhyankar's article [1], p.417, "So the algebraist was freed from the shackles of analysis, or rather (as in Vedanta philosophy) he was told that he was always free but had only forgotten it temporarily."

nilpotent functions. To get a quick glimpse of this concept, for any field K, let $R = K[X]$ or $K[[X]]$ and consider the ring $R_1 = R/(X^2) = K[\epsilon_1]$, where ϵ_1 is the coset of X in R_1. An element $a + b\epsilon_1 \in R_1$, for $a, b \in K$ can be considered infinitesimally close to a of order 1; note that elements $b\epsilon_1$ are infinitesimally close to 0 and may simply be called infinitesimals of first order.

While calculating with infinitesimals, we use $\epsilon_1^2 = 0$, which makes precise the age-old practice of 'neglecting the terms of order 2 or more' for the infinitesimals of order 1. There is a weaker version of cancellation of ϵ_1 in R_1: For $b_1, b_2 \in R_1, b_1\epsilon_1 = b_2\epsilon_1$ implies $b_1 \equiv b_2 \mod \epsilon_1$. As a consequence, we see that $1 + b_1\epsilon_1 = 1 + b_2\epsilon_1$ implies $b_1 \equiv b_2 \mod \epsilon_1$ and hence we get a well-defined map

$$\rho : \{1 + b\epsilon_1 : b \in R_1\} \to R_1/(\epsilon_1)R_1 \cong K$$

given by $\rho(1 + b\epsilon_1) = b + (\epsilon_1)R_1$; the right-hand side can be identified with its representative from K.

Coming to the relevance of infinitesimals to our cause, suppose a, b are approaching the multiplicative identity 1 as $t \to \infty$, for instance, say $a = 1 - \frac{u}{t}, b = 1 - \frac{v}{t}$, where $u = u_0 + u_1 t^{-1} + u_2 t^{-2} + \ldots, v = v_0 + v_1 t^{-1} + v_2 t^{-2} + \ldots$ with u_i, v_i independent of t for all $i = 0, 1, \ldots$. Then, as $t \to \infty$, $u \to u_0, v \to v_0$ while

$$u \otimes_t v = \frac{1}{t^{-1}} \left[(1 - \frac{u}{t})(1 - \frac{v}{t}) - 1\right] \to u_0 + v_0 \ .$$

This can be explained, without using the limits, by reverting to the formal language of infinitesimals of 1st order introduced above as follows: Letting $\epsilon_1 = -t^{-1} \mod t^{-2}$ to be an infinitesimal of first order, we have $(1 + u\epsilon_1)(1 + v\epsilon_1) = 1 + (u + v)\epsilon_1$. Notice that $u_0 = \lim_{t\to\infty} T(a) = \rho(a)$, $v_0 = \lim_{t\to\infty} T(b) = \rho(b)$ and $u_0 + v_0 = \lim_{t\to\infty} T(ab) = \rho(ab)$. Thus the map ρ explains the magical phenomenon we started with.

In effect, it means that by unslanting the auxiliary line, the multiplication of elements infinitesimally near to 1 gets converted to the addition of the corresponding quantities.

Box 2. Marius Sophus Lie

This 19th century Norwegian mathematician lived from 1842 AD to 1899 AD. His work on transformation groups linked several areas in mathematics such as group theory, differential equations and geometry. A whole new set of ideas later became known as Lie theory and were the key to further developments in theoretical physics. His partnership with another 19th century giant Felix Klein is the most significant and productive partnership in mathematics till date.

Lie Groups and their Lie Algebras

The above considerations fall in the realm of Lie groups and their Lie algebras. Lie groups, originally studied by the Norwegian Mathematician Sophus Lie as continuous transformation groups, are now known as those groups which also have analytic manifold structure that is compatible with the group structure, i.e., the multiplication as a binary operation and inverse as a unary operation are analytic maps. The key to studying Lie groups were the associated structures known as Lie algebras, whose elements were called berührungstransformationen (i.e., contact transformations or infinitesimal transformations) by Lie in his context. To understand this association in the modern context, we introduce the following terminology. Let R be the ring of analytic functions on the Lie group G, $\mathbb{R}$ be the field of real numbers and $R(1)$ be the ring of the germs of analytic functions at the identity 1 of G and $M(1)$ be its unique maximal ideal. Let $Der(R)$ denote the set of all $\mathbb{R}$-derivations d of R, i.e., $\mathbb{R}$-linear maps $d : R \to R$ such that $d(ab) = ad(b) + bd(a)\ \forall a, b \in R$ and let $Der^G(R) = \{d \in Der(R) : d(f^g) = d(f)\ \forall f \in R, g \in G\}$ where $f^g(a) = f(ga)\ \forall a \in G$. The Lie algebra $\mathfrak{g}$ of G, as a real vector space, is canonically isomorphic to any of the following (see [3]):

(1) The vector space $Hom_{\mathbb{R}}(M(1)/M(1)^2, \mathbb{R})$ of all $\mathbb{R}$-linear maps from $M(1)/M(1)^2$ to $\mathbb{R}$.
(2) $Der^G(R)$.
(3) $Der_{\mathbb{R}}(R(1), \mathbb{R})$ = The set of all $d : R(1) \to \mathbb{R}$ such that
$d(ab) = ad(b) + bd(a)\ \forall a, b \in R(1)$ and $d(a) = 0\ \forall a \in \mathbb{R}$.

In case $G = \mathbb{R}^+$, the multiplicative group of positive real numbers, $\mathfrak{g}$ is also isomorphic to the multiplicative group $L(G) = \{1 + b\epsilon : b \in \mathbb{R}\} \subset \mathbb{R}[X]/(X^2)$, where $\epsilon = X + (X^2)$; this gives the description of $\mathfrak{g}$ in terms of infinitesimals. See *Box* 3 for an intuitive discussion and p.450 of [10] for a rigorous discussion. On

Box 3. What are Lie Groups and Lie Algebras?

To make the twin structures of Lie groups and their Lie algebras more accessible, we explain them here using special cases, viz. matrix groups such as (1) the general linear group $\mathrm{GL}(n,\mathbb{R})$, which is the multiplicative group of all invertible real $n\times n$ matrices, (2) the special linear group $\mathrm{SL}(n,\mathbb{R})$ of all real $n\times n$ matrices whose determinant is 1, (3) the orthogonal group $O(n)$ of all (real) orthogonal $n\times n$ matrices (i.e., those matrices whose inverse is given by their transpose). Each of these groups is an 'analytic manifold'. This means, roughly speaking, that near each of its element, it looks like an open ball in a Euclidean space and these small Euclidean space-like neighbourhoods are patched smoothly. While $\mathrm{GL}(n,\mathbb{R})$ can itself be considered as an open subset of $\mathbb{R}^{n^2}$, for $\mathrm{SL}(n,\mathbb{R})$ we can explicitly solve the defining equation $\det(A)=0$ locally for one variable (whose cofactor is nonzero) in terms of the rest, identifying that neighbourhood with an open ball in (n^2-1)-dimensional space. In general, we could apply the implicit function theorem of calculus to solve locally the defining implicit equations of groups. In addition to being an analytic manifold, the multiplication and the operation of taking inverses in these groups are analytic maps, when locally viewed as maps between open sets in Euclidean spaces. An analytic manifold which is also a group and whose multiplication and inverse are analytic maps, is called a Lie group.

The implicit function theorem referred to in the above paragraph also leads to the notion of the Lie algebra of a Lie group. The implicit function theorem relates the situation on a manifold near a point, say P, to the similar situation in a related linear space; the process is called *linearization*. Thus, corresponding to arbitrary maps on a manifold near its point P, there are linear maps on the related linear spaces; Implicit function theorem essentially says that if the linear equations on the related linear spaces are solvable, then so are the corresponding equations on the manifolds in a neighbourhood of P. The said linear space, in fact, is *the tangent space* to the manifold at P. It consists of all tangent vectors at P to various curves on the manifold through P. Intuitively, a tangent vector to a curve at a point P may be thought of as the limiting case of a secant vector $\vec{PQ}$ as point Q on the curve approaches P. Now a vector $\vec{A}$ is a secant vector at P to some curve on a manifold if and only if $\lambda\vec{A}=\vec{PQ}$ for some scalar $\lambda\neq 0$ and some point Q on the manifold. Moreover Q is close to P when λ is small. Thus intuitively it is imaginable that $\vec{A}$ is a tangent vector to the manifold at P if and only if P and Q, an infinitesimal perturbation of P in the direction of $\vec{A}$, are both on the manifold. Q is an infinitesimal perturbation of P means that $\vec{q}=\vec{p}+\epsilon\vec{A}$ for an infinitesimal ϵ, where $\vec{p}$ and $\vec{q}$ are the position vectors of P and Q respectively and the addition is in the ambient Euclidean space, the manifold. See *Figure* A.

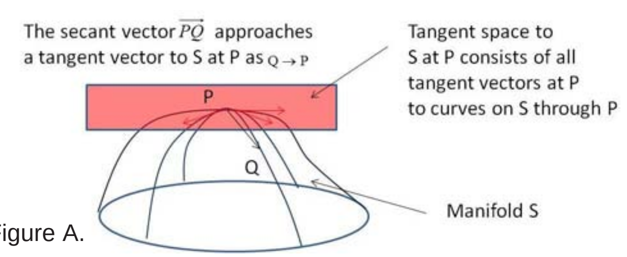


Figure A.

Box 3. Continued...

The Lie algebra $L(G)$ of a Lie group G is nothing but the tangent space of G at its identity element. As $\mathrm{GL}(n,\mathbb{R})$ is an open set in $\mathbb{R}^{n^2}$, its Lie algebra is an n^2-dimensional linear space, which can be identified with the real vector space $M_n(\mathbb{R})$ of all real $n \times n$ matrices. On the other hand, linearizing the defining equation $det(A) = 0$ of $\mathrm{SL}(n,\mathbb{R})$ at its identity matrix (i.e., replacing this equation by the linear equation obtained by the truncation of its power series expansion around the identity matrix), we see that its Lie algebra can be identified with the $n^2 - 1$ dimensional vector space of all matrices of $M_n(\mathbb{R})$ whose trace is 0. Similarly, the defining implicit equations of $O(n)$ are the orthonormality relations between the rows of a matrix, and replacing them by their linearization at the identity matrix, the Lie algebra of $O(n)$ can be identified with the $\left(\frac{n^2-n}{2}\right)$-dimensional vector space of all skew symmetric matrices in $M_n(\mathbb{R})$. In fact there is also a multiplication defined on these vector spaces, making it an $\mathbb{R}$-algebra. This multiplication is called the *Lie bracket*. In all the three Lie algebras considered above, Lie bracket is defined by $[A,B] = AB - BA$. Note that the Lie bracket is *not* associative, but, as a consequence of the associativity of ordinary matrix multiplication, we can see easily that the Lie bracket satisfies the Jacobi identity $[A,[B,C]] + [B,[A,C]] + [C,[A,B]] = 0$. Sometimes a tangent vector at the identity may be identified with an infinitesimal perturbation of identity in its direction; in this case $L(G)$ can be thought of as the collection of all infinitesimal perturbations on G of the identity elements of G and the addition in $L(G)$ would then be thought as multiplication in G, (see the main text for details).

This algebra structure of Lie algebra of a Lie group G encodes many properties of G itself. In fact, the local Lie group structure of G (i.e., a small neighbourhood of identity and multiplication of its elements) can be constructed from just its Lie algebra $L(G)$. The main role in this link is played by the exponential map $\exp : L(G) \to G$. For the matrix groups discussed above, exp is nothing but the matrix exponential. Thus $\exp(A) = \sum_{i=0}^{\infty} \frac{A^i}{i!}$. It can be verified for all the examples of Lie groups G considered above, that $\exp(A) \in G$ if $A \in L(G)$.

The concepts of tangent space, Lie bracket and exponential map become more involved for Lie groups more general than matrix groups. Some of the technicalities are alluded to in the main text. Interested readers may look up the references provided at the end of the article for further details.

Another level of abstraction can be achieved by replacing Lie groups by analytic loops. Analytic loops are defined just like Lie groups except that the associativity of multiplication is not required. If we define the Lie algebra of an analytic loop in a manner similar to Lie groups, then the corresponding Lie bracket may not satisfy the Jacobi identity. If the associative property of the multiplication in Lie group is replaced by the *Moufang property* $z(x(zy)) = ((zx)z)y$ for all $x, y, z \in G$ then it is called an *analytic Moufang loop*. If the Jacobi identity in a Lie Algebra is replaced by the identity $[[x,y],[z,x]] = [[[x,y],z],x] + [[[y,z],x],x] + [[[z,x],x],y]$, then the resulting structure is called a *Lie–Moufang* or a *Malcev* algebra. It was shown by Malcev that the relationship between an analytic Moufang loop and its Lie–Moufang algebra via exponential map works in the same manner as the relationship between a Lie group and its Lie algebra. For further details, see the book by Gorbatsevich, Onishchik, Vinberg [4].

the other hand, analogously it can be seen that the the Lie algebra of the additive Lie group $\mathbb{R}$ can be identified with $L(\mathfrak{g}) = \{b\epsilon : b \in \mathbb{R}\} \subset \mathbb{R}[X]/(X^2)$. Any of these vector spaces can be converted into an algebra by introducing Lie bracket, a binary operation which is in general non-associative. It can easily be defined on $Der^G(R)$ as $[x, y] = xy - yx \; \forall x, y \in Der^G(R)$; note that xy denotes the composition of x, y as maps from R to itself and is not in $Der^G(R)$ in general, though $[x, y]$ always is. The most important notion relating $\mathfrak{g}$ to G is the map $exp : \mathfrak{g} \to G$, called the exponential map. When $G = \mathbb{R}^+$, the multiplicative group of the positive real numbers, $\mathfrak{g} = \mathbb{R}$ and the exponential map coincides with the usual exponential series $e^x = \sum_{i=0}^{\infty} \frac{x^n}{n!}$. This map induces the map exp^*, the derivative of the exponential map at 0, from the $L(\mathfrak{g})$, the tangent space of $\mathfrak{g}$ at 0, to $L(G)$, the tangent space of G at 1, such that $\exp^*(b\epsilon) = 1 + b\epsilon$.

In turn, by composing with the identification of b with $b\epsilon$ for $b \in \mathbb{R}$, we recover the map ρ^{-1}, the inverse of ρ defined and related to unslanting of the auxiliary line in the paragraph on infinitesimals.

3. Conclusion

In this note, we have hinted at the link between two themes connecting the binary operations of addition and multiplication, viz., the ruler constructions and Lie theory. We have done it with a bare exposition of these themes, just enough to arrive at the link. This is due to the shortage of space, time as well as the competence of the author. Still to get some sense of the task at hand, we outline the author's perspective on the literature. For the theme of ruler constructions, we refer to [6], [8]. For the theme of Lie theory, we refer to [5] for a differential geometric approach while [2] or [9] for a more formal and algebraic approach. The formal approach also helps in the study of algebraic groups and p-adic groups, which, in turn, have consequences in algebraic number theory. Lie theory can be extended to a situation when the associative property of the multiplication of the Lie group is relaxed to Moufang property, that is, instead of Lie groups one considers analytic Moufang loops. To make it possible, the concept of Lie algebras needs to be generalized to Lie–Moufang algebras. This extension is discussed in [4], and is recommended for a better understanding of Lie theory. It could be interesting to probe the extent to which this Lie–Moufang correspondence relates to the ruler construction for non-associative multiplication, the later is discussed in [6]. Remarkably, the Moufang property of multiplication was discovered by Ruth Moufang while studying the ruler construction of multiplication in certain non-desarguesian planes. Though the Lie theory connection goes up to Moufang property of multiplication in analytic loops, it is not clear if it can go up to the ruler construction of multiplication, in the spirit of this article.

Suggested Reading

[1] S S Abhyankar, Historical ramblings in algebraic geometry and related algebra, *American Mathematical Monthly*, Vol.83, No.6, 1976, pp.409–448.

[2] N Bourbaki, *Lie Groups and Lie Algebras*, Ch.1–3, Springer, Berlin, 1989.

[3] D Bump, *Lie groups*, Springer, New York, 2004.

[4] V V Gorbatsevich, A L Onishchik, E B Vinberg, *Foundations of Lie Theory and Lie Transformation Groups*, Springer, Berlin, 1997.

[5] S Helgason, Differential geometry, Lie groups, and symmetric spaces. Corrected reprint of the 1978 original, *Graduate Studies in Mathematics*, Vol.34, *American Mathematical Society*, Providence, RI, 2001.

[6] D Hughes and F Piper, *Projective Planes*, Springer Verlag, New York, 1973.

[7] R E Moritz, *Memorabilia Mathematica*, MacMillan, New York, 1914.

[8] A Seidenberg, *Lectures in Projective Geometry*, van Nostrand Reinhold Company, New York, 1962.

[9] J-P Serre, *Lie Algebras and Lie Groups*, Lecture Notes in Mathematics, No.1500, Springer, Berlin, 2006.

[10] M van der Put, Galois theory of differential equations, algebraic groups and Lie algebras. Differential algebra and differential equations, *J. Symbolic Comput.*, Vol.28, No.4–5, pp.441–472, 1999.

Address for Correspondence
P H Keskar
Mathematics Department
Birla Institute of Technology and Science
Pilani 333 031, India.
Email: keskar@bits-pilani.ac.in