How Many Equivalent Resistances?

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It is straightforward to construct the set of equivalent resistance for circuits constructed from a bunch of four or five equal resistors. But as the bunch size increases it becomes difficult to find the order of the set of equivalent resistances. Even the computer programs runs out of memory. Here we present an analytical result using simple mathematical machinery. The size of the set is shown to be less than 2.618^n.

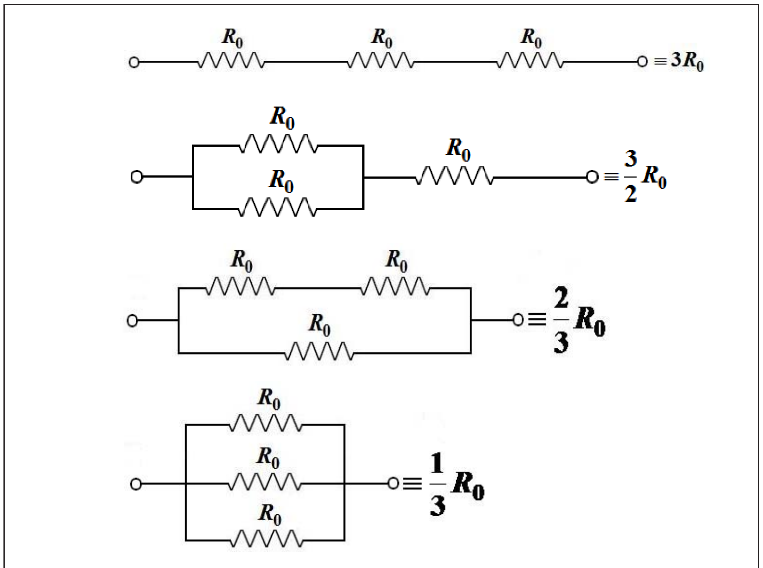
1. Introduction

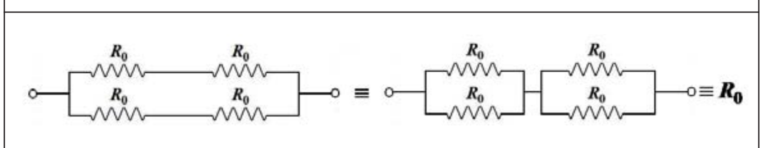
In an introductory physics course one finds exercises such as: *Find all the resistances that can be realized using three equal resistors in various combinations* [1]. The 4 possible solutions are shown in *Figure* 1.

If the exercise is to use *3 or fewer* equal resistors, there are seven solutions; the three additional solutions being R_0 (using one resistor) and $2R_0$ and $(1/2)R_0$ (using two resistors combined in series and parallel respectively). We continue the exercise with 4 resistors and find 9 equivalent resistances, but ten different configurations. The two configurations shown in *Figure* 2 have the same equivalent resistance.

We note that different configurations can give rise to the same equivalent resistance. Next, we analyze the case of five resistors. One possible configuration is the bridge network [2], whose equivalent resistance for equal resistors is R_0 (see *Figure* 3). Using exclusively series and/or parallel combinations, results in 22 equivalent resistances. We shall initially focus on series and parallel combinations, and then consider the case of bridge circuits. The order of the set of equivalent resistances

Keywords
Resistors, series, parallel, bridge, non-planar, Fibonacci numbers, Haros–Farey sequence, integer sequences.





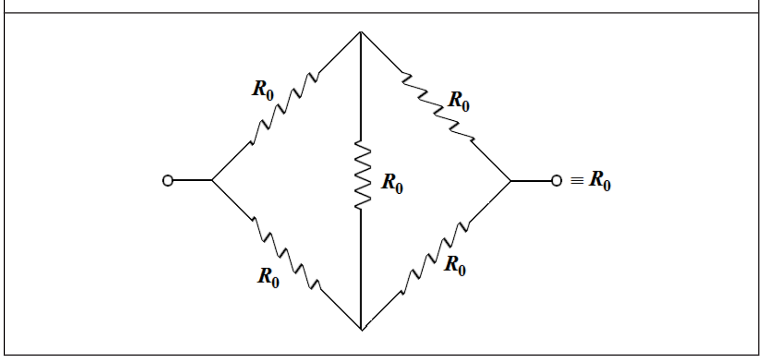


Figure 1 (top). Circuit configurations for 3 resistors.

Figure 2 (centre). Equivalent circuits in the case of 4 resistors.

Figure 3 (bottom). Bridge circuit.

grows rapidly and we have for $n = 1, 2, 3, \ldots, A(n) = 1, 2, 4, 9, 22, 53, 131, 337, 869, 2213, 5691, 14517, 37017, \ldots$, respectively. The problem for n up to 16 has been addressed computationally [3]. We shall cite the various integer sequences occurring in this study, by the unique identity assigned to each of them in *The On-Line Encyclopedia of Integer Sequences* (OEIS), created and maintained by Neil Sloane [4]. For instance the sequence $A(n)$ is identified by A048211 and OEIS has seven additional terms,whereas [3] contains the first 16 terms. The sequence, 1, 2, 4, 10, 24, 66, 180, 522, 1532, 4624, ..., giving the growth of the number of circuit configurations corresponding to the sets $A(n)$, occurs in different contexts such as the number of unlabeled cographs on n nodes [A000084]. The number of configurations is much larger than the number of equivalent resistances.

We shall address the question analytically and provide an upper bound for $A(n)$. The approximate formula, $A(n) \sim 2.53^n$, obtained from the numerical data upto $n = 16$ in [3] is consistent with the analytical result, $A(n) < 2.618^n$ presented here. We shall initially consider the case of using all the n resistors; then extend it to the case of n *or fewer* resistors (i.e., at most n resistors). The key ingredients of the mathematical machinery we shall use are described in *Boxes* 1 and 2.

Box 1. Fibonacci Numbers

Fibonacci numbers are the sequence of numbers, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... [A000045]. Successive Fibonacci numbers are obtained by taking the sum of the two preceding numbers; this is expressed as the linear recurrence relation

$$F_n = F_{n-1} + F_{n-2},$$

for $n \geq 3$, with $F_1 = F_2 = 1$. The linear recurrence relations are solved by introducing a constant ratio, $\lambda = F_n/F_{n-1}$, between any two successive terms. This leads to the quadratic equation $\lambda^2 = \lambda + 1$, with one of the roots as $\phi = (1+\sqrt{5})/2$. The ratios of the pair of Fibonacci numbers, F_{n+1}/F_n rapidly converges to the golden ratio, $\phi = (1+\sqrt{5})/2 = 1.61803398874....$ This ratio occurs in diverse situations and hence has been named as the *golden ratio* or even the *divine proportion*. Computation of an arbitrary Fibonacci number is facilitated by the closed form expression

$$F_n = \left[\frac{\phi^n}{\sqrt{5}}\right],$$

where $[\ldots]$ is the nearest integer function. The lore surrounding the Fibonacci numbers is gigantic and there is even a journal, *The Fibonacci Quarterly,* devoted to the study of integers with special properties, published by 'The Fibonacci Association', (http://www.fq.math.ca/).

Box 2. Haros–Farey Sequence

The Haros–Farey sequence of order m (a natural number) is the set of irreducible rational numbers a/b with $0 \leq a \leq b \leq m$. The fractions are traditionally arranged in order of increasing size [A005728]. The first few are

$$\begin{aligned}
Farey(1) &= 2 : \left\{ m = 1 : \frac{0}{1}, \frac{1}{1} \right\}, \\
Farey(2) &= 3 : \left\{ m = 2 : \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \right\}, \\
Farey(3) &= 5 : \left\{ m = 3 : \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right\}, \\
Farey(4) &= 7 : \left\{ m = 4 : \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right\}, \\
Farey(5) &= 11 : \left\{ m = 5 : \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} \right\}.
\end{aligned}$$

Each Farey sequence starts with zero (denoted by 0/1) and ends with 1 (denoted by 1/1), and contains all conceivable fractions whose denominators are less than or equal to m. The Farey sequence of order m contains all of the members of the Farey sequences of all lower orders. The sequence is generally attributed to Farey (CE 1816), but an earlier publication can be traced to Haros (CE 1802). Hence, the name Haros–Farey sequence. Unlike in the case of Fibonacci numbers, there is no known closed form expression for the Farey sequence. However, the asymptotic limit for Farey(m) is

$$Farey(m) \sim \frac{3}{\pi^2} m^2 + O(m \log m).$$

The number of fractions in the subinterval, $I = [\alpha, \beta]$ is proportional to the length of the subinterval, $|I| = |\beta - \alpha| = (\beta - \alpha)$ by the relation

$$Farey(m; I) = |I| \times Farey(m) = 3(\beta - \alpha) \frac{1}{\pi^2} m^2.$$

2. Results and Analysis

Let R_0 be the value of the n equal resistors being used. The net resistance of all the configurations is proportional to the unit resistance R_0; this unit resistance can be set to unity without any loss of generality. The proportionality constant is a rational number (say a/b; with a and b being natural numbers, a/b is in its reduced form) depending on the configuration. The value of a/b

ranges from $1/n$ (for all the n resistors in parallel configuration) to n (for all the n resistors in series configuration). For the series and/or parallel connections, the set of values of a/b, for the first few n are

$$\begin{aligned}
A(1) &= 1 : \{n = 1 : 1\}, \\
A(2) &= 2 : \left\{n = 2 : \frac{1}{2}, 2\right\}, \\
A(3) &= 4 : \left\{n = 3 : \frac{1}{3}, \frac{2}{3}, \frac{3}{2}, 3\right\}, \\
A(4) &= 9 : \left\{n = 4 : \frac{1}{4}, \frac{2}{5}, \frac{3}{5}, \frac{3}{4}, 1, \frac{4}{3}, \frac{5}{3}, \frac{5}{2}, 4\right\}, \\
A(5) &= 22 : \left\{n = 5 : \frac{1}{5}, \frac{2}{7}, \frac{3}{8}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{5}{8}, \frac{5}{7}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \right. \\
& \qquad \left. \frac{7}{6}, \frac{6}{5}, \frac{5}{4}, \frac{7}{5}, \frac{8}{5}, \frac{7}{4}, 2, \frac{7}{3}, \frac{8}{3}, \frac{7}{2}, 5\right\}.
\end{aligned}$$

Throughout, we shall use the same symbols, $A(n)$, $B(n)$, etc., to denote sets and their order respectively. In the above sets, we make two observations. Each value of a/b in a given set occurs in a reciprocal-pair of a/b and b/a respectively; 1 being its own partner (see [3] for a proof by induction). The largest value of a and b in a given set is equal to F_{n+1}, the $(n+1)$th term in the Fibonacci sequence[5–6, A000045]; this largest value is obtained for the ladder network [1–2]. A set $A(n)$ of higher order does not necessarily contain the complete sets of lower orders. For example, 2/3 is present in the set $A(3)$, but it is not present in the sets $A(4)$ and $A(5)$.

By virtue of the reciprocal relation, it suffices to count the number of equivalent resistances less than 1. The values of a and b are bounded by F_{n+1}. So, the problem of deriving $A(n)$ translates to counting the number of 'relevant' proper fractions, whose denominators are bounded by $m = F_{n+1}$. Farey sequence provides the most exhaustive set of fractions in the interval [0, 1], whose denominators are less than or equal to a given natural number m [5, A005728]. The set of proper fractions in the set for $A(n)$ is bounded in the subinterval I = [$1/n$, 1], which is a subinterval of [0, 1]. Recall that the resistance, $1/n$, is obtained by taking all the n resistors in parallel. The length of the interval I is |I| = ($1 - 1/n$). Considering that all elements of $A(n)$ have a reciprocal pair except 1, and the fact that 1 is included in the Farey sequence (1 gets counted twice); we have the expression

$$G(n) = 2Farey(m; I) - 1 = 2Farey\,(F_{n+1}; I) - 1. \quad (1)$$

$G(n)$ by construction is the grand set (superset) containing the fractions from $Farey\,(F_{n+1})$ in the interval [1/n, 1] along with their reciprocals [A176502]. The set $G(n)$ contains all rational numbers of the form a/b such that both a and b are bounded by F_{n+1}. Since the Farey sequence is exhaustive, the set $G(n)$ is also exhaustive. This leads to the strict upper bound

$$A(n) < G(n) = 2Farey\,(F_{n+1}; I) - 1. \quad (2)$$

Ignoring the –1 in the above expression, and using the asymptotic relation for Farey(m) and the closed form expression for F_{n+1}, we have

$$\begin{aligned} G(n) \sim 2 \quad &\times \quad |I|\frac{3}{\pi^2}m^2 \sim (1-\frac{1}{n})\frac{6}{\pi^2}\left(\frac{\phi^{n+1}}{\sqrt{5}}\right)^2 \\ &= \quad (1-\frac{1}{n})\frac{6\phi^2}{5\pi^2}\phi^{2n} \\ &= \quad (1-\frac{1}{n})0.318 \times (2.618)^n, \end{aligned} \quad (3)$$

where ϕ is the *golden ratio* [5]. The approximate formula, $A(n) \sim 2.53^n$, obtained from the numerical computations up to $n = 16$ in [3] and $n = 23$ in [A048211] is consistent with the analytical results presented above. The asymptotic formula for $G(n) \sim 2.618^n$ strictly fixes the upper bound of $A(n)$. When using $G(n)$ for $A(n)$, Farey sequence is the most exhaustive set of fractions, so it is sure to contain some terms absent in the actual circuit configurations.

there is a certain amount of over counting. Farey sequence is the most exhaustive set of fractions, so it is sure to contain some terms absent in the actual circuit configurations.

When we go beyond the series and parallel configurations (such as the bridge circuits), the Farey scheme is still applicable in providing a strict upper bound. The bridge circuits respect the bound theorem [6]. Hence, the Farey scheme becomes applicable to the bridge circuits (even in the absence of the reciprocal theorem). Hence, all equivalent resistances of configurations containing bridge circuits belong to the grand set $G(n)$. The set $B(n)$ containing bridge circuits (in addition to the configurations produced by series and/or parallel; the set $A(n)$ is completely contained in $B(n)$) has the strict bounds

$$A(n) < B(n) < G(n) = 2Farey\,(F_{n+1}; I) - 1. \tag{4}$$

The sets $A(n)$ are for the restricted case of using all the n resistors. The Farey sequence framework is applicable to the scenario of relaxing the restriction to n *or less* resistors. Let $C(n)$ denote the total number of equivalent resistances obtained using one or more of the n equal resistors; the order of the sets are 1, 3, 7, 15, 35, 77, 179, 429, 1039, 2525, … [A153588]. The set $C(n)$ is the union of all the sets $A(i)$, $i = 1, 2, 3, \ldots\ n$,

When we go beyond the series and parallel configurations (such as the bridge circuits), the Farey scheme is still applicable in providing a strict upper bound.

$$C(n) = \bigcup_{i=1}^{n} A(i). \tag{5}$$

It is to be recalled that the sets $A(i)$ can have elements which may not be present in the set $A(j)$, where $j \neq i$. Each $A(i)$ is obtained from $Farey\,(F_{i+1})$. Farey sequence of a given order contains all the members of the Farey sequences of all lower orders. So, the set $C(n)$ is strictly bounded by the Farey scheme and we have

$$A(n) < C(n) < G(n) = 2Farey\,(F_{n+1}; I) - 1. \tag{6}$$

3. Concluding Remarks

We have addressed the question of the order of the set of equivalent resistances $A(n)$ of n equal resistors combined in series and in parallel analytically, a topic traditionally approached computationally. The upper bound is fixed by $G(n)$, the grand set, constructed using the Haros–Farey sequence with the Fibonacci numbers as its argument. The approximate formula, $A(n) \sim 2.53^n$, obtained from the computational data up to $n = 23$ is consistent with the analytical strict upper bound, $A(n) < 2.618^n$ presented here. It is further shown that the Farey sequence approach, developed for the $A(n)$ is applicable to configurations other than the series and/or parallel, namely the bridge circuits and non-planar circuits; bounds for such circuit configurations are also presented. The scheme also enables us to understand the case of $C(n)$, the order of the set of equivalent resistances using *n or fewer* equal resistors. A comprehensive account of the resistor problem with computer programs is available in [6].

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Suggested Reading

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