Unidirectional Motion of Vehicle on Sinusoidal Path

Can it Cause Illusory Forward and Backward Motion?

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The feeling of being in motion, while observing a moving train passing nearby is a common experience. It is an example of relative motion where a stationary observer experiences an illusory motion caused by a vehicle moving in the opposite direction. It is however counter-intuitive to imagine periodic changes in the direction of relative motion, while the vehicle continues to move in one direction only. To an observer, sitting inside the moving vehicle, periodic changes in direction can lead to the illusion of being in oscillatory forward and backward relative motion. This article, using a simple mathematical model describes how the unidirectional motion of vehicle on a sinusoidal path can lead to the illusion of being in periodic forward and backward relative motion. This work has emerged out of a real life experience during a train journey on the hilly curves from Kalka to Shimla.

Every object in the universe, be it a subatomic particle or a galaxy, is moving relative to the rest of the objects. The importance of relative motion can be best understood with the help of the special theory of relativity. The direction and magnitude of relative motion between two inertial frames can lead to the same event being interpreted differently by observers in their respective frames. At times, we encounter situations where relative motion can cause illusory observation and make us believe what is actually not true. The study of such a situation is presented here.

Keywords
Relative velocity, sinusoidal path, periodic motion.

1. Vehicle Moving on a Sinusoidal Path

Consider a vehicle moving with constant speed V on a sinusoidal path whose axis is curved like a circle, as shown in *Figure* 1. A digital camera facing the center O of the path is attached to the side window of the vehicle. The observer sitting inside the vehicle has the freedom to look outside in any direction. If R is the radius of the circle, A the amplitude of sinusoidal path and θ the angle of rotation, the radial distance $r(\theta)$ between the vehicle and the fixed reference O is determined by

$$r(\theta) = R + A\cos\theta_r\,, \qquad (1)$$

where $\theta_r = \frac{2\pi\theta}{\theta_0}$; θ_0 defines the periodicity of the sinusoidal path and is chosen to be 18^o in our study.

The position of the vehicle, $p(\theta)$, at any point on the curved path can be expressed in vector form as

$$\vec{p}(\theta) = r(\theta)\cos\theta\,\hat{\imath} + r(\theta)\sin\theta\hat{\jmath}\,, \qquad (2)$$

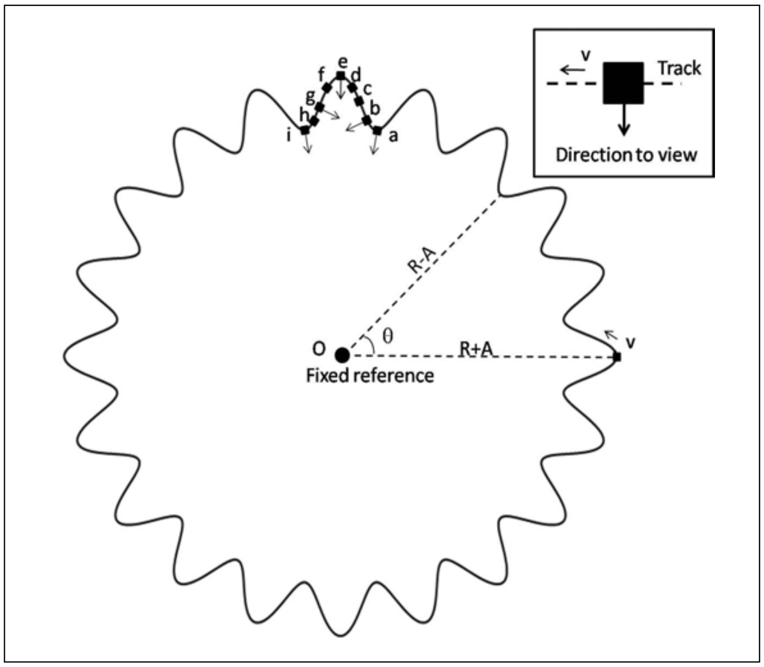


Figure 1. The schematic shows a sinusoidal path with its axis curved to form a circle. A small black square representing the vehicle moves on this curved path. The center O of the circle can be viewed from any given point on the shown path and, act as a fixed reference for the observer sitting inside the vehicle. The vehicle moves anti-clockwise at a constant speed V and makes an angle θ with the x axis.

where $\hat{\imath}$ and $\hat{\jmath}$ are the unit vectors along the x and y axes respectively. The direction of motion of the vehicle can be determined by drawing a tangent to the curve and expressed as

$$\begin{aligned}\dot{\vec{p}}(\theta) &= \frac{\mathrm{d}}{\mathrm{d}\theta}\vec{p}(\theta) \\ &= P\,\hat{\imath} + Q\hat{\jmath}\,, \end{aligned} \tag{3}$$

where

$$P = -\left[r(\theta)\sin\theta + \frac{2\pi A}{\theta_0}\sin\theta_r\cos\theta\right]\,, \tag{4}$$

$$Q = \left[r(\theta)\cos\theta - \frac{2\pi A}{\theta_0}\sin\theta_r\sin\theta\right]\,. \tag{5}$$

For a given angle θ, a unit vector centered at O and pointing towards the moving vehicle is described as

$$\hat{u} = \cos\theta\,\hat{\imath} + \sin\theta\hat{\jmath}\,. \tag{6}$$

The angle between the unit vector $\hat{u}$ and the direction of motion $\dot{\vec{p}}(\theta)$ is determined by

$$\dot{\vec{p}}(\theta)\cdot\hat{u} = |\dot{p}(\theta)|\,|\hat{u}|\cos\alpha\,.$$

That is,

$$\alpha(\theta) = \cos^{-1}\left(\frac{[P\cos\theta + Q\sin\theta]}{\sqrt{P^2+Q^2}}\right) \tag{7}$$

because $|\hat{u}| = 1$.

The camera is fixed to view at right angle to the direction of motion, as shown in the inset of *Figure* 1. This arrangement makes the camera swing periodically from $-\phi_{\max}$ to $+\phi_{\max}$; $+\phi_{\max}$ to $-\phi_{\max}$; $-\phi_{\max}$ to $+\phi_{\max}$ and so on, as the vehicle moves along the curved path. Here, ϕ is the angle between the camera-view direction and the radial vector, and is given by

$$\phi(\theta) = \alpha(\theta) - 90^{\circ}\,. \tag{8}$$

The vehicle in which she is travelling will appear to be periodically moving forward and reverse, without realizing that actually she is moving at a constant velocity in one direction only.

It is also the angle of deflection of the camera with respect to the fixed reference O.

The periodic change in ϕ between $-\phi_{\max}$ to $+\phi_{\max}$ as a function of θ has an illusory effect on the observer who is inside the vehicle and constantly looking outside in the direction of the camera. To the observer, the vehicle in which she is travelling will appear to be periodically moving forward and reverse, without realizing that actually she is moving at a constant velocity in one direction only.

This effect can be best explained by calculating the radial position of the vehicle and angular deflection ϕ as a function of θ, for positions a to i of *Figure* 1. These results are shown in *Figure* 2. The solid line represents the position of the vehicle with respect to the axis of the oscillatory path. It varies periodically as a function of θ between $-A$ and $+A$, where A is taken as 10 m. The angle of deflection at a, e and i is zero and maximum at c and g. The plot can be divided into regions of increasing and decreasing ϕ, as shown by alternating dark and white vertical strips in *Figure* 2.

The illusory periodic motion effect can be weak or strong depending upon the shape and size of the path. The

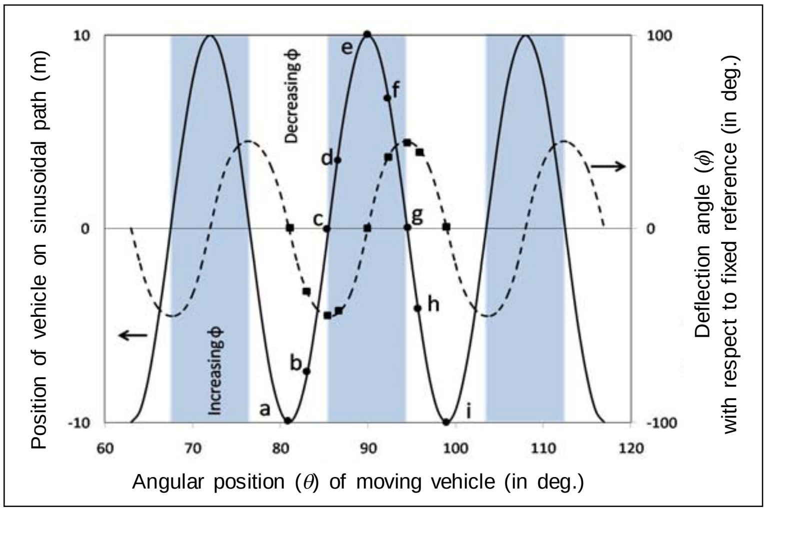


Figure 2. The movement of vehicle on a small section of sinusoidal path extending from angular position $\theta = 60^{\circ}$ to 120° is shown. The corresponding angle of deflection ϕ relative to fixed reference O is also traced.

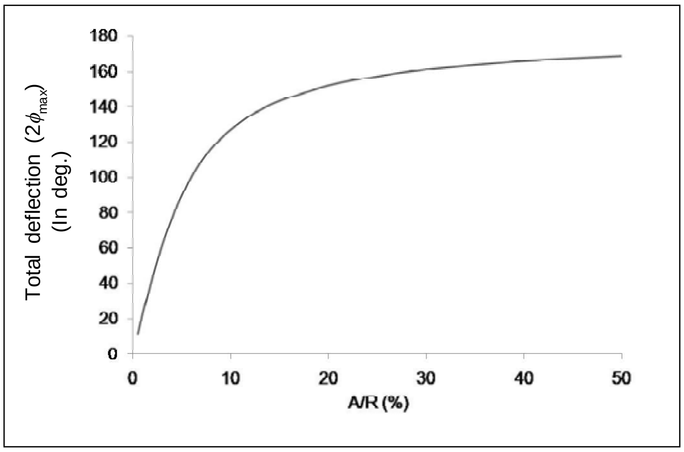


Figure 3. The ratio of sinusoidal path amplitude *A* and radius *R* of circular path determines the total angle of deflection $2\phi_{max}$. The depicted plot is for θ_o =18°.

variation in total angle of deflection $2\phi_{\max}$ as a function of A/R is shown in *Figure* 3. The observer will hardly notice any periodic motion for deflection angle less than 15° at $A/R \sim 1\%$; this can be called weak effect. As A/R increases, the corresponding deflection angle also increases, leading to noticeable periodic motion and hence the strong effect. The illusory periodic motion is clearly visible to the observer under such circumstances. The deflection angle increases rapidly at smaller A/R values and increases rather slowly when A/R is more than 20 %.

It may not be easy to imagine how periodic changes in the deflection angle can make the observer feel like moving forward and reverse periodically. To aid in this, pictures of a background hill were taken while the vehicle was moving on a curved path, as shown in *Figure* 4. The snaps correspond to vehicle positions a to i. The highest point of the hill is taken as a reference and marked with an arrow pointing downwards. From a to c, the deflection angle ϕ decreases making the hilltop shift towards right and making the observer feel as though she is moving left (direction of horizontal arrow in picture). From c to g, the deflection ϕ increases making the hilltop shift towards left and leaving an impression as though the observer has started moving right. Again,

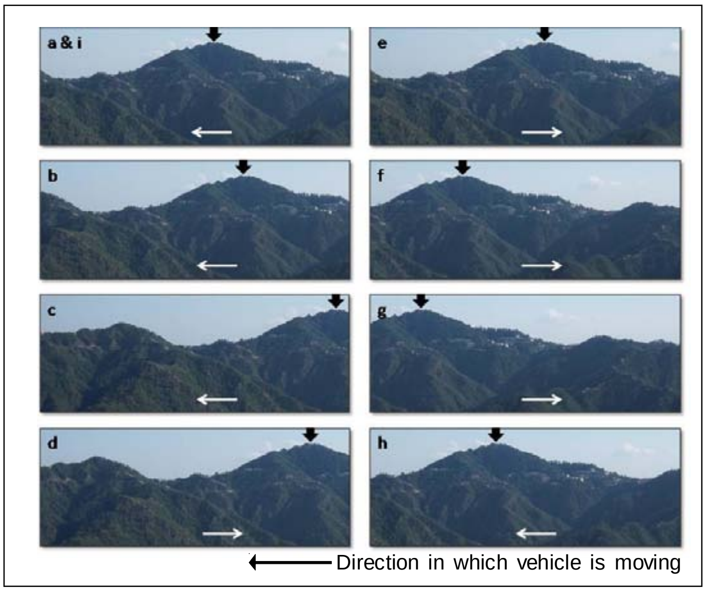
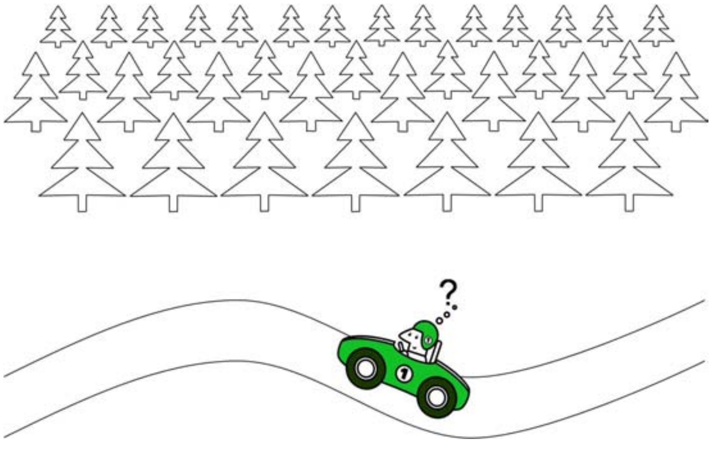


Figure 4. The unidirectional movement (anticlockwise in this case) of vehicle on sinusoidal path, and stationary background as seen by observer sitting inside the moving vehicle is shown. Pictures *a* to *i* of stationary background correspond to vehicle positions *a* to *i* of *Figure* 2. The vertical arrow pointing towards the peak of the hill is the point of reference. The position of this arrow shifts to right until *c*, then starts moving to left until *g* before turning to right again. The horizontal arrow in each picture indicates the direction of relative motion experienced by the observer due to shift in reference point.

g to i is the region of decreasing ϕ where hilltop shifts to the right and the observer gets the feeling of moving left. This illusory effect can be observed provided the observer looks constantly in the direction of the camera while the vehicle keeps moving at constant speed on a curved path.



***Driving fast but not heading anywhere. It appears as if I am moving forward-reverse-forward-…….reverse. Any clue, what is happening*?**

Suggested Reading

[1] **Robert Resnick, *Introduction to special relativity*, John Wiley Sons Inc., USA, 1968.**

[2] **Murray R Spiegel, *Theory and problems of vector analysis*, Schaum's outline series, McGraw-Hill, USA, 1959.**