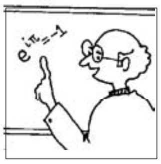
# Classroom



**In this section of *Resonance,* we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.**

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## A Train Journey between Two Terminating Stations

**A train undertakes a journey between two terminating stations with a given distance apart and halts at some intermediate stations. It travels between two consecutive intermediate stations of this kind with uniform acceleration, uniform velocity and then with uniform retardation. Neglecting the air resistance, the optimization problem herein is to find the optimal distances of these intermediate stations so as to minimize the total time of travel by the train, given the durations of uniform velocity.**

**Keywords**
Optimization, Lagrange multipliers.

### 1. Formulation of the Problem

Let a train undertake a journey between two terminating stations $A_1$ and $A_{n+1}$, a distance $S$ apart. It starts from rest from station $A_1$ with uniform acceleration $f_1$ and stops at the next station $A_2$ for some time. The train stops at each intermediate station for sometime. In between any two consecutive intermediate stations, say, $A_i$ and $A_{i+1}$ it begins to travel with uniform acceleration $f_i$ from rest and attains a certain speed $v_i$ after sometime. Thereafter it maintains this speed for some time $\tau_i$ and applies a brake generating uniform retardation $f_i$ to stop

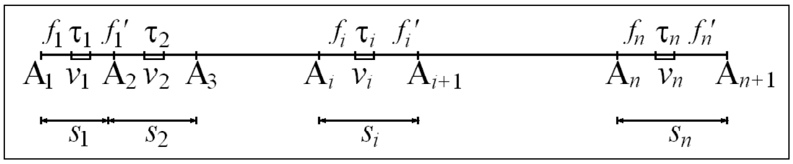


**Figure 1.**

at the next station $A_{i+1}$. The problem is to determine the optimum distances of the intermediate stations for given values of $\tau_i$, $S, f_i$ (denoting acceleration) and $f_i'$ (denoting retardation) so that the train completes the total journey in minimum time, excluding the haltage times at those stations (vide *Figure* 1).

## 2. Solution to the Problem

Let $s_i$ be the distance travelled by the train in time $t_i$ between the $i$th and $(i+1)$th intermediate stations during which it attains a maximum velocity $v_i$. Then we get

$$t_i = v_i\left(\frac{1}{f_1}+\frac{1}{f_1'}\right)+\tau_i\,, \tag{1}$$

$$s_i = \frac{v_i^2}{2}\left(\frac{1}{f_1}+\frac{1}{f_1'}\right)+v_i\tau_i\,. \tag{2}$$

The total time of travel and distance between two terminating stations are given by summing (1) and (2) over $i$.

$$T = \Sigma_{i=1}^n t_i = \sum_{i=1}^{n}\left\{v_i\left(\frac{1}{f_i}+\frac{1}{f_i'}\right)+\tau_i\right\}\,, \tag{3}$$

$$S = \sum_{i=1}^{n} s_i\,. \tag{4}$$

Since the distance between the terminating stations is fixed, we have to minimize $T$ subject to the constraint (4) with respect to $v_i$. This will give us the optimum values of $v_i$ from which we shall get optimum values of $s_i$. Using Lagrange's multiplier $\lambda$ for this, one gets

$$F(s_i) = T+\lambda S^* = \sum_{i=1}^{n}(v_i a_i+\tau_i)+\lambda\left(S-\sum_{i=1}^{n}s_i\right)\,, \tag{5}$$

where

$$a_i = \frac{1}{f_i} + \frac{1}{f'_i}\ , \quad S^* = S - \sum_{i=1}^{n} s_i\,. \qquad (6)$$

Solving the quadratic equation (2) for $v_i$ and substituting in (3), (4) and (5) followed by use of (6), we get

$$v_i = \frac{1}{a_i}\left(-\tau_i + \sqrt{\tau_i^2 + 2s_i a_i}\right)\ , \qquad (7)$$

$$T = \sum_{i=1}^{n}\left\{\sqrt{\tau_i^2 + 2s_i a_i}\right\}\ , \qquad (8)$$

(discarding the negative value of $v_i$). Therefore

$$F(s_i) = \sum_{i=1}^{n}\left(\sqrt{\tau_i^2 + 2s_i a_i}\right) + \lambda\left(S - \sum_{i=1}^{n} s_i\right)\ . \qquad (9)$$

For minimum value of $T$ subject to the constraint (4),

$$\frac{\partial F(s_i)}{\partial s_i} = \frac{a_i}{\sqrt{\tau_i^2 + 2s_i a_i}}\ - \lambda = 0, \quad (i = 1, 2, 3, \ldots)\,. \qquad (10)$$

That is,

$$\tau_i^2 + 2s_i a_i = \frac{a_i^2}{\lambda^2}\ ,$$

or,

$$2s_i = \frac{a_1}{\lambda^2} - \frac{\tau_i^2}{a_i}\ . \qquad (11)$$

Summing over $i = 1, 2, 3, \ldots n$, using (4), equation (11) gives

$$2S = \frac{1}{\lambda^2}\sum_{i=1}^{n} a_i - \sum_{i=1}^{n}\frac{\tau_i^2}{a_i}$$

or,

$$\frac{1}{\lambda^2} = \left[2S + \sum_{i=1}^{n}\tau_i^2/a_i\right]\Big/\sum_{i=1}^{n} a_i\,. \qquad (12)$$

Substituting this in (11) we have the optimum distance between the $i$th and $(i+1)$th stations as

$$(s_i)_{\text{opt}} = \left[ a_i \left\{ S + \frac{1}{2} \sum_{i=1}^{n} \frac{\tau_i^2}{a_i} \right\} \right] \Big/ \sum_{i=1}^{n} a_i - \frac{\tau_i^2}{2a_i} . \quad (13)$$

Likewise the other two optimum parameters are obtained using (2), (6), (7), (1) and (13):

$$(t_i)_{\text{opt}} = a_i \sqrt{2 \left\{ S + \frac{1}{2} \sum_{i=1}^{n} \frac{\tau_i^2}{a_i} \right\} \Big/ \sum_{i=1}^{n} a_i} \, , \quad (14)$$

$$(v_i)_{\text{opt}} = \frac{-\tau_i}{a_i} + \sqrt{2 \left\{ S + \frac{1}{2} \sum_{i=1}^{n} \frac{\tau_i^2}{a_i} \right\} \Big/ \sum_{i=1}^{n} a_i} \, . \quad (15)$$

Finally summation of (14) yields the minimum time of travel by the train excluding the haltage times, from rest to rest between the two terminuses

$$\begin{aligned} T_{\min} &= \sum_{i=1}^{n} \left[ a_i \sqrt{ \left\{ 2S + \sum_{i=1}^{n} \frac{\tau_i^2}{a_i} \right\} / \sum_{i=1}^{n} a_i } \right] \\ &= \left[ \left( \sum_{i=1}^{n} a_i \right) \left( 2S + \sum_{i=1}^{n} \frac{\tau_i^2}{a_i} \right) \right]^{1/2} . \quad (16) \end{aligned}$$

Examining (13), summation $\sum (s_i)_{\text{opt}}$ obviously becomes $S$. Equations (13) and (14) give the optimum time of reaching the $(i+1)$th intermediate station and its distance as

$$T_j = \sum_{i=1}^{j} (t_i)_{\text{opt}} = \sum_{i=1}^{j} \left[ a_i \left\{ \left( 2S + \sum_{i=1}^{n} \frac{\tau_i^2}{a_i} \right) \Big/ \sum_{i=1}^{n} a_i \right\}^{1/2} \right] \quad (17)$$

$$S_j = \sum_{i=1}^{j} \left[ a_i \left( S + \frac{1}{2} \sum_{i=1}^{n} \frac{\tau_i^2}{a_i} \right) \Big/ \sum_{i=1}^{n} a_i - \frac{\tau_i^2}{2a_i} \right] . \quad (18)$$

The foregoing equations (13) to (16) can be rewritten as

$$(s_i)_{\rm opt} = \frac{1}{2a_i}\left\{(t_i)^2_{\rm opt} - \tau_i^2\right\}\ , \qquad (19)$$

$$(t_i)_{\rm opt} = (a_i)T_{\rm min}\Big/\sum_{i=1}^{n} a_i\ , \qquad (20)$$

$$(v_i)_{\rm opt} = \frac{1}{a_i}\left\{(t_i)_{\rm opt} - \tau_i\right\}\ , \qquad (21)$$

$$T_{\rm min} = \sqrt{\left(\sum_{i=1}^{n} a_i\right)\left(2S + \sum_{i=1}^{n}\frac{\tau_i^2}{a_i}\right)}\ . \qquad (22)$$

To corroborate the kinematics developed, we present a numerical example here.

## 3. Numerical Example

Let the train commute between two terminating stations $A_1$ and $A_5$, which are 250 km apart with three intermediate stations $A_2$, $A_3$, $A_4$ so that $S = 250$ km, $n = 3 + 1 = 4$, and also let us take $\tau_1 = 4$ minutes $= \frac{1}{15}$ hr, $\tau_2 = 6$ minutes $= \frac{1}{10}$ hr, $\tau_3 = 8$ minutes $= \frac{2}{15}$ hr, $\tau_4 = 10$ minutes $= \frac{1}{6}$ hr, as durations of 'uniform speeds'. Let the uniform accelerations be $f_1 = 80$ km/hr$^2$, $f_2 = 120$ km/hr$^2$, $f_3 = 160$ km/hr$^2$, $f_4 = 200$ km/hr$^2$, and the uniform decelerations (retardations) be $f_1' = 400$ km/hr$^2$, $f_2' = 480$ km/hr$^2$, $f_3' = 600$ km/hr$^2$, $f_4' = 640$ km/hr$^2$. Then, using the first part of relation (6) we get

$$\begin{aligned}
a_1 &= \frac{1}{4}\left(\frac{1}{20} + \frac{1}{100}\right) = \frac{1}{4}\times\frac{3}{50} = \frac{3}{200}\ ({\rm km/hr})^{-1}\\
a_2 &= \frac{1}{4}\left(\frac{1}{30} + \frac{1}{120}\right) = \frac{1}{4}\times\frac{1}{24} = \frac{1}{96}\\
a_3 &= \frac{1}{4}\left(\frac{1}{40} + \frac{1}{150}\right) = \frac{1}{4}\times\frac{19}{600} = \frac{19}{2400}\\
a_4 &= \frac{1}{4}\left(\frac{1}{50} + \frac{1}{160}\right) = \frac{1}{4}\times\frac{21}{800} = \frac{21}{3201} \qquad (23)
\end{aligned}$$

Using (22), the minimum time $T_{\min}$ of travel is

$$\begin{aligned} T_{\min} &= \left[2 \times 250 + 4\left\{\left(\frac{1}{15}\right)^2 \times \frac{3}{50} + \left(\frac{1}{10}\right)^2 \times 24 \right.\right. \\ &\quad \left.\left. + \left(\frac{1}{6}\right)^2 \times \left(\frac{600}{19}\right) + \left(\frac{2}{15}\right)^2 \times \left(\frac{800}{21}\right)\right\}\right] \\ &\quad \times \left[\frac{1}{4}\left(\frac{3}{50} + \frac{1}{24} + \frac{19}{600} + \frac{21}{800}\right)\right]^{1/2} \\ &= \left(507.5 \times \frac{383}{2400 \times 4}\right)^{1/2} = 4.49 \simeq 4.5 \text{ hr} \quad (24) \end{aligned}$$

which in consequence of (20) gives the time of journey between two consecutive stations:

$$\begin{aligned} (t_1)_{\text{opt}} &= \frac{(a_1 T_{\min})}{\sum a_i} = \frac{1}{4} \times \frac{3}{50} \times (4.5) \times \frac{4 \times 2400}{383} \\ &= 1.692 \text{ hr} = 1 \text{ hr } 41 \text{ mt } 32 \text{ secs}, \\ (t_2)_{\text{opt}} &= \frac{1}{4} \times \frac{1}{24} \times \frac{9}{2} \times \frac{4 \times 2400}{383} = 1.175 \text{ hr} \\ &= 1 \text{ hr } 10 \text{ mts } 32 \text{ secs}, \\ (t_3)_{\text{opt}} &= \frac{600}{4 \times 19} \times \frac{9}{2} \times \frac{4 \times 2400}{383} = 0.893 \\ &= 53 \text{ mts } 32 \text{ secs}, \\ (t_4)_{\text{opt}} &= \frac{1}{4} \times \frac{800}{21} \times \frac{9}{2} \times \frac{4 \times 2400}{383} = 0.740 \\ &= 44 \text{ mts } 26 \text{ secs}. \end{aligned}$$

The optimum distances between two consecutive stations required to minimize the total time of travel are given by (using (19))

$$\begin{aligned} (s_1)_{\text{opt}} &= \frac{1}{2} \times \frac{4 \times 50}{3}\left\{(1.692)^2 - \left(\frac{1}{15}\right)^2\right\} = \frac{285.84}{3} \\ &= 95.22 \text{ km}, \\ (s_2)_{\text{opt}} &= \frac{1}{2} \times 4 \times 24\left\{(1.175)^2 - \left(\frac{1}{10}^2\right)\right\} \\ &= 66.27 \text{ km}, \end{aligned}$$

$$
\begin{aligned}
(s_3)_{\text{opt}} &= \frac{1}{2} \times \frac{4 \times 600}{19} \left\{ (0.893)^2 - \left( \frac{2}{15} \right)^2 \right\} \\
&= 49.11 \text{ km}, \\
(s_4)_{\text{opt}} &= \frac{1}{2} \times \frac{4 \times 800}{21} \left\{ (0.740)^2 - \left( \frac{1}{6} \right)^2 \right\} \\
&= 39.40 \text{ km}.
\end{aligned}
$$

The maximum speeds reached in between two consecutive stations are obtained from (21) as

$$
\begin{aligned}
v_1 &= \frac{200}{3} \left( 1.692 - \frac{1}{15} \right) = 108.35 \text{ km/hr}, \\
v_2 &= 96 \ (1.175 - 0.1) = 103.35 \text{ km/hr}, \\
v_3 &= \frac{2400}{21} \left( 0.893 - \frac{2}{15} \right) = 95.95 \text{ km/hr}, \\
v_4 &= \frac{3200}{21} \left( 0.740 - \frac{1}{6} \right) = 87.38 \text{ km/hr}.
\end{aligned}
$$

It is observed that the train runs with an average speed of $250/4.5 = 55.56$ km/hr. Note that while calculating $s_i$ and $t_i$ $(i = 1, 2, 3, 4)$, fractions are rounded off so as to find their summations as 250 km and 4.5 hr respectively.

So, we find that to obtain minimum time of travel between two terminal stations 250 km apart, the optimum distances of the intermediate stations are 95.22, 161.49 and 210.60 km respectively.

## 4. Discussion

If the air resistance [1] proportional to the velocity or square of the velocity is taken into consideration, in order to travel with uniform velocity for sometime, the force exerted by the engine must be equal to the air resistance and rail friction. Hence a relevant model of train travel with air resistance against its motion in conformity with the present feature can be prepared and solved. Relation (13) reveals that the optimal spacings of the intermediate stations, $(s_i)_{\text{opt}}$, are not equal but dependent on $a_i$ (i.e., the uniform acceleration and the uniform retardation) and also duration $\tau_i$ of uniform motion but will be equal for equal values of $a_i$ and of $\tau_i$. It would be of interest to speculate about a somewhat different model by replacing uniform acceleration $f_i$ by uniform horse power in the present problem.

In the present investigation, a question may arise as to what will be the speed of the train sighted at a distance, say, $S_0$ from the originating station $A_1$ or vice-versa. Its answer is as follows; vide *Figure* 2.

If the train has already crossed $p$ intermediate stations, in the above position, then

$$S_0 = \sum_{i=1}^{p} s_i + s_0 \ , \qquad (25)$$

where $s_0$ is the distance of the train from the intermediate station $p$ on acquiring the velocity $v_0$; $v_p$ being the maximum velocity between $A_p$ and $A_{p+1}$.

Case 1. If

$$0 < s_0 < \frac{v_p^2}{2f_p}\ , \text{ then } v_0 = (2f_p s_0)^{1/2}. \qquad (26)$$

Case 2. If

$$\frac{v_p^2}{2f_p} < s_0 < \frac{v_p^2}{2f_p} + v_p\tau_p\ , \text{ then } v_0 = v_p\,. \qquad (27)$$

Case 3. If

$$\frac{v_p^2}{2f_p} + v_p\tau_p < s_0 < \frac{v_p^2}{2f_p} + v_p\tau_p + \frac{v_p^2}{2f_p'}\ , \text{ then}$$

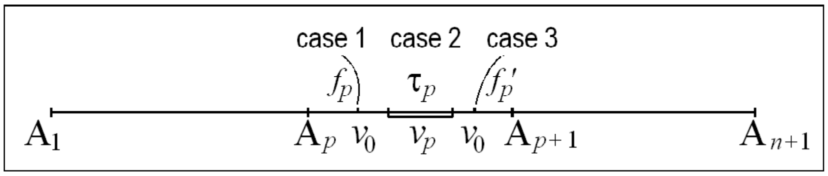


Figure 2.

$$v_0 \quad = \quad \left[2f_p'\left(\frac{v_p^2}{2f_p}+v_p\tau_p+\frac{v_p^2}{2f_p'}-s_0\right)\right]^{1/2} . \qquad (28)$$

Suppose one wishes to find the velocity after some time, say $T_0$, given by

$$T_0 = \sum_{i=1}^{p} t_i + t_0 \,. \qquad (29)$$

This suggests replacement of $s_0$ by $\frac{1}{2}f_p t_0^2$. Then

$$v_0 = f_p t_0 \text{ when } 0 < t_0 < \frac{v_p}{f_p};$$

$$v_0 = v_p \text{ when } \frac{v_p}{f_p} + \tau_p > t_0 > \frac{v_p}{f_p};$$

$$v_0 = \left(\frac{v_p}{f_p} + \frac{v_p}{f_p'} + \tau_p - t_0\right) f_p'$$

$$\text{when } \frac{v_p}{f_p} + \tau_p < t_0 < \frac{v_p}{f_p} + \frac{v_p}{f_p'} + \tau_p \;.$$

## Suggested Reading


**[1] S N Maitra, Minimum Time of Travel, *Resonance*, Vol. 7, No.7, pp.80–83, 2002.**

**[2] G C Sharma and M Ray, *A textbook on Dynamics for BA/BSc students*, S Chand and Co. Ltd., New Delhi, pp.45–47, 2000.**