Ancient Indian Mathematics – A Conspectus*

S G Dani

S G Dani is a Distinguished Professor at the Tata Institute of Fundamental Research, Mumbai. He obtained his bachelor's, master's and PhD degrees from the University of Mumbai. His areas of interest are dynamics and ergodic theory of flows on homogeneous spaces, probability measures on Lie groups and history of mathematics. He has received several awards including the Ramanujan Medal and the TWAS Prize.

* This is a slightly modified version of an article by the author published in *Ramanujan@125*, a special pullout brought out by *The Hindu* on 26 December 2011.

Keywords
Sulvasutras, Jain mathematics, *Siddhanta* astronomy, *Bakhshali* manuscript, Kerala School of Mathematics.

India has had a long tradition of more than 3000 years of pursuit of Mathematical ideas, starting from the Vedic age. The *Sulvasutras* (which included Pythagoras theorem before Pythagoras), the Jain works, the base 10 representation (along with the use of 0), names given to powers of 10 up to 10^{53}, the works of medieval mathematicians motivated by astronomical studies, and finally the contributions of the Kerala school that came strikingly close to modern mathematics, represent the various levels of intellectual attainment.

There is now increasing awareness around the world that as one of the ancient cultures, India has contributed substantially to the global scientific development in many spheres, and mathematics has been one of the recognized areas in this respect. The country has witnessed steady mathematical developments over most part of the last 3,000 years, throwing up many interesting mathematical ideas well ahead of their appearance elsewhere in the world, though at times they lagged behind, especially in the recent centuries. Here are some episodes from the fascinating story that forms a rich fabric of the sustained intellectual endeavour.

1. Vedic Knowledge

The mathematical tradition in India goes back at least to the *Vedas*. For compositions with a broad scope covering all aspects of life, spiritual as well as secular, the *Vedas* show a great fascination for large numbers. As the transmission of the knowledge was oral the numbers were not written, but they are expressed as combinations of powers of 10, and it would be reasonable to believe

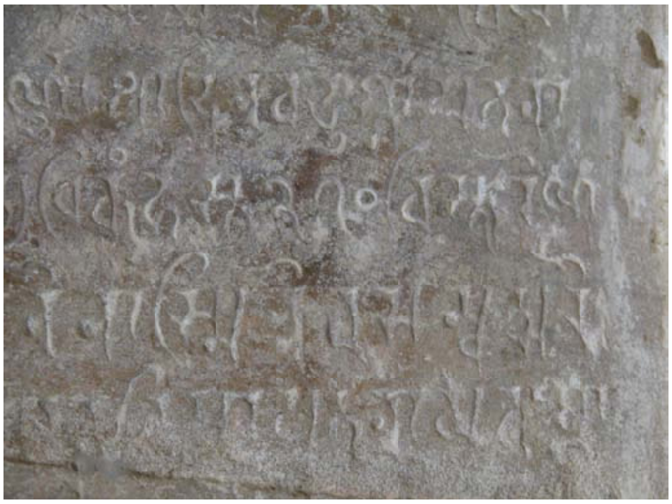

Figure 1. Portion of a dedication tablet in a rock-cut Vishnu temple in Gwalior built in 876 AD. The number 270 seen in the inscription features the oldest extant zero in India.
Courtesy: Bill Casselman

that when the decimal place value system for written numbers came into being it owed a great deal to the way numbers were discussed in the older compositions.

The decimal place value system of writing numbers, together with the use of 0, is known to have blossomed in India in the early centuries AD, and spread to the west through the intermediacy of the Persians and Arabs. There were actually precursors to the system, and various components of it are found in other ancient cultures such as the Babylonian, Chinese, and Mayan. From the decimal representation of the natural numbers the system was to evolve further into the form that is now commonplace and crucial in various walks of life, with decimal fractions becoming part of the number system in 16th century Europe, though this again has some intermediate history involving the Arabs. Evolution of the number system represents a major phase in the development of mathematical ideas, and arguably contributed greatly to the overall advance of science and technology. The cumulative history of the number system holds a lesson that progress of ideas is an inclusive phenomenon, and while contributing to the process should be a matter of joy and pride to those with allegiance to the respective contributors, role of others also ought to be appreciated.

Emergence and evolution of zero as a mathematical entity has a complex and multi-layered history, involving several cultures, in terms of various aspects of it such as the base level, place holder in representing numbers, mathematical symbol, a number on its own, etc.

It is well known that Geometry was pursued in India in the context of construction of *vedis*[1] for the *yajnas* of the Vedic period. The *Sulvasutras*[2] contain elaborate descriptions of constructions of *vedis* and also enunciate various geometric principles. These were composed in the first millennium BC, the earliest *Baudhayana Sulvasutra* dating back to about 800 BC. The *Sulvasutra* geometry did not go very far in comparison to the Euclidean geometry developed by the Greeks, who appeared on the scene a little later, in the seventh century BC. It was however an important stage of development in India too. The *Sulvasutra* geometers were aware, among other things, of what is now called the Pythagoras theorem, over two hundred years before Pythagoras (all the four major *Sulvasutras* contain explicit statement of the theorem), addressed (within the framework of their geometry) issues such as finding a circle with the same area as a square and vice versa, and worked out a very good approximation to the square root of 2, in the course of their studies.

Though it is generally not recognised, the *Sulvasutra* geometry was itself evolving. This is seen in particular

[1] The Vedic culture had the unique feature of performance of fire rituals, known as *yajnas*. These involved well laid out altars, the *vedis*, and fire platforms, known as *citi* or *agni*, elaborately constructed in the form of birds, tortoise, wheel, etc.

[2] The *Sulvasutras* are compositions, in the form of manuals for construction of *vedis* and *citis*, but they also discuss the geometric principles involved.

Box 1. What is Vedic Mathematics?

The term 'Vedic mathematics' has also come to carry in recent times another connotation than what is involved here. It is associated with the book by that name, first published in 1965, authored by Shri Bharati Krishna Tirthaji. The book describes short-cut procedures for certain kinds of arithmetical or algebraic computations, to which have been associated short phrases in Sanskrit, serving as *aide-memoire* for the individual procedures; the book refers to 16 of these phrases as '*sutras*', and another assortment of similar phrases as '*sub-sutras*'. These phrases have no basis in the Vedic literature, and are extraneous to the *Vedas* in various respects including language, style and mathematical content. The mathematics described in the book is essentially a twentieth century formulation, and it is just that the author calls it "Vedic mathematics". We shall not go into a critique of the system here (for which the reader may refer to the present author's articles on the topic available online). It may be emphasized however that it should not be confused with the mathematics from the Vedic period which is one of the themes of the present article.

from the differences in the contents of the four major extant *Sulvasutras*. Certain revisions are especially striking. For instance, in the early *Sulvasutra* period the ratio of the circumference to the diameter was, like in other ancient cultures, thought to be 3, as seen in a *sutra* of Baudhayana, but in the *Manava Sulvasutra* a new value was proposed, as 3 and one-fifth; interestingly the *sutra* describing it ends with an exultation "not a hair-breadth remains", and though we see that it is still substantially off the mark, it is a gratifying instance of an advance made. In the *Manava Sulvasutra* one also finds an improvement over the method described by Baudhayana for finding the circle with the same area as that of a given square.

2. Jain Tradition

The Jain tradition has also been very important in the mathematical development in the country. Unlike for the Vedic people, for the Jain scholars the motivation for mathematics came not from ritual practices, which indeed were anathema to them, but from contemplation of the cosmos. The Jains had an elaborate cosmography in which mathematics played an integral role, and even largely philosophical Jain works are seen to incorporate mathematical discussions. Notable among the topics in the early Jain works, from about the 5th century BC to 2nd century AD, one may mention geometry of the circle, arithmetic of numbers with large powers of 10, permutations and combinations, and categorisations of infinities (whose plurality had been recognised).

In the Jain work *Suryaprajnapti* from the fourth century BC after recalling the then "traditional" value 3 for the ratio of the circum-ference of the circle to its diameter, the author discards it in favour of the square root of 10.

As in the *Sulvasutra* tradition the Jains also recognised, around the middle of the first millennium BC that the ratio of the circumference of the circle to its diameter is not 3. In *Suryaprajnapti*, a Jain text believed to be from the 4th century BC, after recalling the "traditional" value 3 for it the author discards that in favour the square root of 10. This value for the ratio, which

While Pythagoras theorem is not mentioned specifically in the early Jain literature, the formulae involving the circular arcs and chords clearly suggest awareness of the theorem.

is reasonably close to the actual value, was prevalent in India over a long period and is often referred as the Jain value. It continued to be used long after Aryabhata introduced the well-known value 3.1416 for the ratio. The Jain texts also contain rather unique formulae for lengths of circular arcs in terms of the length of the corresponding chord and the bow (height) over the chord, and also for the area of regions subtended by circular arcs together with their chords. The means for accurate determination of these quantities became available only after the advent of the Calculus. How the ancient Jain scholars arrived at these formulae, which are close approximations, is something remains to be understood.

After a lull of a few centuries in the early part of the first millennium, pronounced mathematical activity is seen again in the Jain tradition from the 8th century until the middle of the 14th century. *Ganitasarasangraha* of Mahavira written in 850, is one of the well-known and influential works. Virasena (8th century), Sridhara (between 850 and 950), Nemicandra (around 980), Thakkura Pheru (14th century) are some more names that may be mentioned. By the 13th and 14th centuries Islamic architecture had taken root in India and in *Ganitasarakaumudi* of Thakkura Pheru, who served as Treasurer in the court of the Khilji Sultans in Delhi, one sees a combination of the native Jain tradition together with Indo-Persian literature, including work on calculation of areas and volumes involved in the construction of domes, arches, and tents used for residential purposes.

3. Astronomy Inspired Mathematics

The mathematical astronomy or what is known as the *Siddhanta* tradition[3] has been the dominant and enduring mathematical tradition in India. It flourished almost continuously for over 7 centuries, starting with Aryabhata (476–550) who is regarded as the founder of scientific astronomy in India, and extending to Bhaskara

[3] In the early centuries of the first millennium great interest arose in India, presumably under Greek influence, and it took firm shape with the work of Aryabhata and Varahamihira by the fifth and sixth centuries. Astronomy was presented in the form of doctrines, *Siddhantas*, and the tradition that emerged from it, involving also considerable development of mathematics along the way, came to be known as the *Siddhanta* tradition.

II (1114–1185), and beyond. The essential continuity of the tradition can be seen from the long list of prominent names that follow Aryabhata, spread over centuries: Varahamihira in the sixth century, Bhaskara I and Brahmagupta in the seventh century, Govindaswami and Sankaranarayana in the ninth century, Aryabhata II and Vijayanandi in the tenth century, Sripati in the eleventh century, Brahmadeva and Bhaskara II in the twelfth century, and Narayana Pandit and Ganesa from the 14th and 16th centuries respectively.

Aryabhata's procedure for finding square roots and cube roots of numbers is in terms of the decimal place value system of representing numbers, which in particular establishes the system being commonly in use during his time.

The *Aryabhatiya*, written in 499, is basic to the tradition, and even to the later works of the Kerala school of Madhava about which I will say more later. It consists of 121 verses divided into four chapters Gitikapada, Ganitapada, Kalakriyapada and Golapada. The first, which sets out the cosmology contains also a verse describing a table of 24 sine differences at intervals of 225 minutes of arc. The second chapter, as the name suggests, is devoted to mathematics per se, and includes in particular procedures for finding square roots and cube roots, an approximate expression for π (amounting to 3.1416 and specified to be approximate), formulae for areas and volumes of various geometric figures, and shadows, formulae for sums of consecutive integers, sums of squares, sums of cubes and computation of interest. The other two chapters are concerned with astronomy, dealing with distances and relative motions of planets, eclipses, etc..

4. Influential Work

Aryabhata also introduced another concise system for expressing numbers, which was used especially for large numbers involved in astrono-mical studies.

Brahmagupta's *Brahmasphutasiddhanta* is a voluminous work, especially for its time, on *Siddhanta* astronomy in which there are also two chapters, the 12th and 18th, devoted to general mathematics. Incidentally Chapter 11 is a critique on earlier works including *Aryabhatiya*; like in other healthy scientific communities this tradition also had many, and often bitter, controversies. Chapter 12 is well known for its systematic treatment of arithmetic operations, including with negative numbers; the notion of negative numbers eluded Europe until the middle of the second millennium. The chapter also contains geometry, including in particular his famous formula for the area of a quadrilateral (stated without the condition of cyclicity of the quadrilateral that is needed for its validity – a point criticised by later mathematicians in the tradition.) The eighteenth chapter is devoted to the *kuttaka* and other methods, including for solving second-degree indeterminate equations. An identity described in the work features also in some current studies where it is referred as the Brahmagupta identity. Apart from this the 21st chapter has verses dealing with trigonometry. The *Brahmasphutasiddhanta* considerably influenced mathematics in the Arab world, and in turn the later developments in Europe.

Bhaskara II is the author of the famous mathematical texts *Lilavati* and *Bijaganita*. Apart from being an accomplished mathematician he was a great teacher and populariser of mathematics. *Lilavati* which literally means 'playful' presents mathematics in a playful way, with several verses directly addressing a pretty young woman, and examples presented through reference to various animals, trees, ornaments, etc. (Legend has it that the book is named after his daughter after her wedding failed to materialise on account of an accident with the clock, but there is no historical evidence to that effect.). The book presents apart from various introductory aspects of arithmetic, geometry of triangles and quadrilaterals, examples of applications of the Pythagoras theorem, *trirasika*[4], *kuttaka*[5] methods, problems on permutations and combinations, etc. The *Bijaganita* is an advanced level treatise on Algebra, the first independent work of its kind in Indian tradition. Operations with unknowns, *kuttaka* and *chakravala*[6] methods for solutions of indeterminate equations are some of the topics discussed, together with examples. Bhaskara's work

[4] *Trirasika* are calculations by 'rule of thumb' or proportionality, in different arithmetical contexts.

[5] *Kuttaka*, or the 'pulverizer', are recursive procedures introduced for solving certain Diophantine equations.

[6] *Chakravala* is a 'cyclical' procedure introduced for solving a class of Diophantine equations which includes the so-called Pell's equation.

on astronomy, *Siddhantasiromani* and *Karana kutuhala*, contain several important results in trigonometry, and also some ideas of Calculus.

The works in the Siddhanta tradition have been edited on substantial scale and there are various commentaries available, including many from the earlier centuries, and works by European authors such as Colebrook, and many Indian authors including Sudhakara Dvivedi, Kuppanna Sastri and K V Sarma. The 2-volume book of Datta and Singh [1] and the book of Saraswati Amma [2] serve as convenient references for many results known in this tradition. Various details have been described, with a comprehensive discussion, in the recent book of Kim Plofker [3].

5. An Enigmatic Manuscript

The *Bakhshali* manuscript, which consists of 70 folios of *bhurjapatra* (birch bark), is another work of crucial significance in the study of ancient Indian mathematics, with many open issues around it. The manuscript was found buried in a field near Peshawar, by a farmer, in 1881. It was acquired by the Indologist A F R Hoernle who studied it and published a short account on it. He later presented the manuscript to the Bodleian library at Oxford, where it has been since then. Facsimile copies of all the folios were brought out by Kaye in 1927, which has since then been the source material for

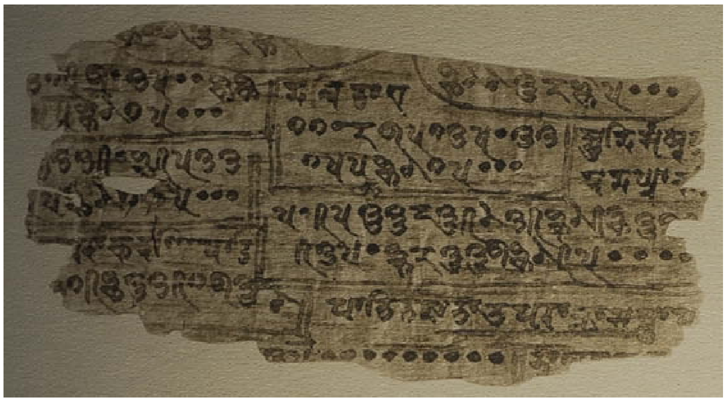

Figure 2. A folio from the *Bakhshali* manuscipt, copied from G R Kaye's 1930 edition. In the middle are two large numbers, with some part on the right missing, forming numerator and denominator of a fraction. It is part of a verification and the fraction is reconstructed to be

$$\frac{50{,}753{,}383{,}762{,}746{,}743{,}271{,}936}{7{,}250{,}483{,}394{,}675{,}000{,}000}$$

Courtesy: Bill Casselman

the subsequent studies. The date of the manuscript has been a subject of much controversy since the early years, with the estimated dates ranging from the early centuries of the first millenium to the 12th century. Takao Hayashi who produced what is perhaps the most authoritative account so far, concludes that the manuscript may be assigned some time between the 8th and the 12th century, while the mathematical work in it may most probably be from the 7th century. Carbon dating of the manuscript could settle the issue, but efforts towards this have not materialised so far.

A formula for extraction of square-roots of non-square numbers found in the manuscript has attracted much attention. Another interesting feature of the *Bakhshali* manuscript is that it involves calculations with large numbers (in decimal representation).

6. Kerala School

Let me finally come to what is called the Kerala School. In the 1830s Charles Whish, an English civil servant in the Madras establishment of the East India Company, brought to light a collection of manuscripts from a mathematical school that flourished in the central part of Kerala, between the present Kozikode and Kochi. The school, with a long teacher-student lineage, lasted for over two hundred years, from the late 14th century well into the 17th century. It is seen to have originated with Madhava, who has been attributed by his successors many results presented in their texts. Apart from Madhava, Nilakantha Somayaji was another leading personality from the School. There are no extant works of Madhava on mathematics (though some works on astronomy are known). Nilakantha authored a book called *Tantrasangraha* (in Sanskrit) in 1500 AD. There have also been expositions and commentaries by many other exponents from the school, notable among them being *Yuktidipika* and *Kriyakramakari* by Sankara and

Mathematics of the Kerala school, while being on the one hand a continuation of the Aryabhatan tradition, on the other hand touches modern mathematics in terms of its treatment of the series for π and trigonometric functions, and the ideas of Calculus involved in it.

Ganitayuktibhasha by Jyeshthadeva which is in Malayalam. Since the middle of the 20th century various Indian scholars have researched on these manuscripts and the contents of most of the manuscripts have been looked into. An edited translation of the latter was produced by K V Sarma [4] and it has recently been published with explanatory notes by K Ramasubramanian, M D Srinivas and M S Sriram. An edited translation of Tantrasangraha has been brought out more recently by K Ramasubramanian and M S Sriram [5].

While mathematical texts in Prakrit have been known from the earlier centuries, *Ganitayuktibhasha* is the earliest known mathematical work to have been composed in any of the present-day Indian languages.

The Kerala works contain mathematics at a considerably advanced level than the earlier works, anywhere around the world. They include a series expansion for π and the arc-tangent series, and the series for sine and cosine functions that were obtained in Europe by Gregory, Leibnitz and Newton, respectively, over two hundred years later. Some numerical values for π that are accurate to 11 decimals are also a highlight of the work. In many ways the work of the Kerala mathematicians anticipated the Calculus as it developed in Europe later, and in particular involves manipulations with indefinitely small quantities (in the determination of circumference of the circle, etc.) analogous to the infinitesimals in Calculus; it has also been argued by some authors that the work is indeed Calculus already.

7. Epilogue

A lot needs to be done to honour this rich mathematical heritage. The extant manuscripts need to be cared for to prevent deterioration, catalogued properly with due updates and, most importantly, they need to be studied diligently and the findings placed in proper context on the broad canvass of the world of mathematics, from an objective standpoint. Let the occasion of the 125th centenary anniversary of the legendary genius of Srinivasa Ramanujan, a global mathematician to the core, inspire us, as a nation, to earnestly apply ourselves to this task.

Suggested Reading

There are many books and articles on the subject. Here are some select titles suggested for further reading on the topic. Further references can be found from the bibliographies in these sources:

[1] B Datta and A N Singh, *History of Hindu Mathematics: A source book*, Parts 1 and 2 (single volume), Asia Publishing House, Bombay, 1962.

[2] Saraswati Amma, *Geometry in Ancient and Medieval India*, Motilal Banarasidas, Delhi, 1979.

[3] Kim Plofker, *Mathematics in India*, Princeton University Press, Princeton, NJ, 2009.

[4] K V Sarma (Ed.), Ganita-yukti-bhasha (Rationales in mathematical astronomy) of Jyesthadeva. Vol.I. Mathematics. Vol. I. Astronomy, with explanatory notes by K. Ramasubramanian, M D Srinivas and M S Sriram, *Sources and Studies in the History of Mathematics and Physical Sciences*, Springer, New York and Hindustan Book Agency, New Delhi, 2008.

[5] K Ramasubramanian and M S Sriram (Ed.), *Tantrasangraha of Nilakantha Somayaji*. With a foreword by B V Subbarayappa, *Culture and History of Mathematics*, Vol.6, Hindustan Book Agency, New Delhi, 2011.

[6] S G Dani, Geometry in *Sulvasutras*, in: *Studies in the history of Indian mathematics*, pp.9–37, Cult. Hist. Math. 5, Hindustan Book Agency, New Delhi, 2010.

[7] George G Joseph, *Crest of the Peacock, Non-European roots of mathematics*, Third edition, Princeton University Press, Princeton, NJ, 2011.

[8] K Ramasubramanian and M D Srinivas, Development of Calculus in India, in: *Studies in the history of Indian mathematics*, pp.201–286, *Cult. Hist. Math.* 5, Hindustan Book Agency, New Delhi, 2010.

Address for Correspondence
S G Dani
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Mumbai 400 005, India.
Email:dani@math.tifr.res.in