Ehrenfest's Theorem and Nonclassical States of Light

2. Dynamics of Nonclassical States of Light

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The states of radiation of a single frequency and polarization propagating in free space are very conveniently represented by those of a quantum mechanical simple harmonic oscillator. This fact is exploited, together with the Ehrenfest theorem, to map the time evolution of the radiation to the dynamics of the oscillator. This enables a graphical comparison of the behaviour of classical and nonclassical states of radiation.

1. Introduction

In the first part[1] of this article, we have introduced Ehrenfest's theorem and discussed its role as a bridge between classical mechanics (CM) and quantum mechanics (QM). In this second part, we shall use the example of the states of a single-mode electromagnetic field to illustrate the theorem in action under various circumstances. In particular, a specific representation for these states will help us relate them to the states of a quantum mechanical oscillator. In turn, this will enable us to present a pictorial description of the evolution of the probability density concerned.

Several interesting states of radiation have been proposed and studied in quantum optics. The experimental realization of these states is a continuing endeavour. Many of these states exhibit properties which have no classical counterparts. In what follows we shall deal in particular, with the departures from 'classicality' exhibited by certain quantum states of radiation. For this purpose, we begin with a brief pedagogical account of how radiation is described in quantum mechanics.

2. Quantum States of the Radiation Field

Quantum optics is concerned with the description and behaviour of various states of light (or the quantized electromagnetic field) as it interacts with atomic media. For the sake of simplicity, we restrict our attention to electromagnetic waves of a given wave vector **k** and state of polarization – either left- or right-circularly polarized. The frequency ω of the radiation is known in terms of its wave vector. For radiation propagating in free space, this relation is just $\omega = c|\mathbf{k}|$. Quantization of the electromagnetic field shows that light of a given wave vector and state of polarization can be treated as a superposition of photon-number states or 'Fock states' $|n\rangle$, where $n = 0, 1, 2, \ldots$ *ad infinitum.* As the name implies, the number of photons present in the state $|n\rangle$ is precisely n. Moreover – and this is crucial –

- *there is a one-to-one correspondence between the set* $\{|n\rangle\}$ *of these Fock states and the familiar set of normalized eigenstates of the Hamiltonian of the linear harmonic oscillator.*

- Quantization of the electromagnetic field therefore amounts to considering it as a (continuously infinite) collection of quantum mechanical linear harmonic oscillators, *one oscillator for each wave vector and polarization state.*

This sort of relationship is typical of field quantization. The underlying reason for it is that each quantum of a 'free' field adds the same amount (in the present case, $\hbar\omega$) to the total energy of the system; and the harmonic oscillator, too, has a discrete spectrum with a constant spacing between adjacent energy levels.

The foregoing is admittedly less than a thumbnail sketch of field quantization, but it suffices for our present purposes. It is important to remember that the set of eigenstates of the Hamiltonian of a *single* linear harmonic

[1] Ehrenfest's Theorem and Non-classical States of Light, 1. Ehrenfest's Theorem in Quantum Mechanics, *Resonance*, Vol.17, No.1, pp.23–32, 2012.

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oscillator is equivalent to the set of photon number states of radiation of a given wave vector and polarization: the latter comprises *multi*-photon states of every non-negative integer number n of photons. We shall exploit this equivalence to visualize the evolution of states of the radiation field in terms of the behaviour of quantum mechanical wave packets (obtained by appropriate superpositions of Fock states) in a harmonic oscillator potential.

Recall that the action of the the lowering and raising operators $\hat{a}$ and $\hat{a}^\dagger$ in the oscillator problem on the state $|n\rangle$ is given by

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle \quad \text{and} \quad \hat{a}^\dagger\,|n\rangle = \sqrt{(n+1)}\,|n+1\rangle, \tag{1}$$

respectively. In the case of the radiation field, $\hat{a}$ is the *photon annihilation operator* while $\hat{a}^\dagger$ is the *photon creation operator* (for photons of a given frequency and state of polarization). They satisfy the basic commutation relation $[\hat{a}\,,\,\hat{a}^\dagger] = \hat{I}$. The set $\{|n\rangle\}$ forms an orthonormal basis in the Hilbert space. Fock states are of course the eigenstates of the *photon number operator*, according to $(\hat{a}^\dagger\,\hat{a})\,|n\rangle = n\,|n\rangle$, where $n = 0, 1, \ldots$. A crucial property is the annihilation of the vacuum or zero-photon state by $\hat{a}$, given by the relation $\hat{a}\,|0\rangle = 0$.

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As we have mentioned already, the Fock states $|n\rangle$ can be superposed to form various interesting states of the radiation field. The states we will consider here are the families of standard coherent states, photon-added coherent states, and squeezed states. These are not merely mathematical constructs. Coherent states represent ideal single-mode laser light, and the one-photon-added coherent state has been produced in the laboratory a few years ago. Squeezing of light has also been demonstrated experimentally. It is expected that propagating beams of squeezed light and multi-photon-added coherent states will be achieved in practice in the near future.

2.1 *Coherent States*

The coherent state (CS) is of basic importance in quantum optics. Let α be any complex number. Then the normalized CS labelled by α is given by

$$\begin{aligned} |\alpha\rangle &= e^{-\frac{1}{2}|\alpha|^2}\left(|0\rangle + \frac{\alpha}{\sqrt{1!}}\,|1\rangle + \frac{\alpha^2}{\sqrt{2!}}\,|2\rangle + \dots\right) \\ &= e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle. \qquad (2) \end{aligned}$$

The CS $|\alpha\rangle$ is a normalized eigenstate of the photon annihilation operator $\hat{a}$.

The exponential factor outside the summation sign ensures that the state is normalized to unity, i.e., $\langle\alpha|\alpha\rangle = 1$. Coherent states have a number of very remarkable properties. The foremost of these is the fact that the CS $|\alpha\rangle$ is a normalized eigenstate of the photon annihilation operator $\hat{a}$. Using the first equation in (1), it is easy to verify that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, and hence $\langle\alpha|\hat{a}^\dagger = \alpha^*\langle\alpha|$, for every complex number α. The eigenvalue spectrum of the nonhermitian operator $\hat{a}$ is thus a two-fold continuous infinity, namely, complex numbers with all possible real and imaginary parts. The set of states $\{|\alpha\rangle\}$ forms what is known as an over-complete set of states in the Hilbert space. Different coherent states are not orthogonal to each other. The mean number of photons in a CS is of significance. This is easily seen to be given by $\langle\alpha|\hat{a}^\dagger\,\hat{a}|\alpha\rangle = |\alpha|^2$. The mean energy of the state $|\alpha\rangle$ is thus $\hbar\omega|\alpha|^2$. The Fock state $|0\rangle$, which is the zero-photon state or the vacuum, is also a coherent state, automatically.

The probability distribution of the number of photons in a CS is also easily determined. By the basic rule of quantum mechanics, the probability that the CS $|\alpha\rangle$ has r photons is found to be $P(r) = |\langle r|\alpha\rangle|^2 = e^{-|\alpha|^2}\,|\alpha|^{2r}/r!$, where $r = 0, 1, 2, \dots$. Hence $P(r)$ is a Poisson distribution, with mean $|\alpha|^2$ (as we have found already). Since the variance is equal to the mean for a Poisson distribu-

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bution, the variance of the photon number in the CS $|\alpha\rangle$ is also equal to $|\alpha|^2$. This is a characteristic signature of the coherent radiation from an ideal single-mode laser. When $|\alpha|$ is sufficiently small, the probability $P(r)$ drops quite rapidly with increasing r. It is therefore possible to arrange matters in a laser experiment such that no photons are produced most of the time, interspersed with production of a single photon now and then. This enables single photon experiments to be carried out.

2.2 *Photon-Added Coherent States*

The m-photon-added coherent state (PACS) $|\alpha, m\rangle$ is obtained by adding m photons to the CS $|\alpha\rangle$ by the repeated application of the photon creation operator $\hat{a}^\dagger$, and then normalizing the resulting state to unity. Thus $|\alpha, m\rangle = C_m\, \hat{a}^{\dagger\, m}\, |\alpha\rangle$, where C_m is a normalization constant. (We shall not write down the exact expression here.) The importance of the PACS arises from several factors, among which are the following:

(a) The state $|\alpha, m\rangle$ interpolates between the Fock state $|m\rangle$ (to which it reduces when $\alpha = 0$) and the CS $|\alpha\rangle$ (to which it reduces when $m = 0$).

(b) The CS $|\alpha\rangle$ enjoys a certain property of perfect coherence, from which the PACS $|\alpha, m\rangle$ departs in a tunable manner as m is increased, keeping α fixed. We do not go into this any further, as we shall not be concerned here with this aspect.

(c) The variance of the photon number in a PACS increases with the mean photon number, but less rapidly than linearly. This leads to a quantifiable departure from Poisson statistics. Again, this is a feature which we do not consider any further here.

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(d) The PACS displays specific nonclassical features that the CS does not. This is the aspect of interest to us here.

As we have stated earlier, the state corresponding to $m = 1$ has been realized experimentally.

Recall that the mean photon number in the CS $|\alpha\rangle$ is $|\alpha|^2$, while it is m in the Fock state $|m\rangle$. At first sight, therefore, we might expect the mean photon number in the PACS $|\alpha, m\rangle$ to be $|\alpha|^2 + m$. The actual answer, however, is a little more involved. It lies between $|\alpha|^2 + m$ and $|\alpha|^2 + 2m$. For a given value of the non-negative integer m, the mean photon number in the PACS starts at m for $|\alpha| = 0$, and tends to $|\alpha|^2 + 2m$ for $|\alpha| \gg m$.

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The third kind of state we shall consider, namely, squeezed states, will be introduced further on.

3. The $\hat{x}$ and $\hat{p}$ Quadratures

In basic courses on QM, the problem of the linear harmonic oscillator is solved in the position representation. The stationary states $|n\rangle$ are then represented by the position-space wave functions $\phi_n(x) \equiv \langle x|n\rangle$. As is well known, the normalized eigenfunction $\phi_n(x)$ is given by the product of a Gaussian factor and the Hermite polynomial of order n. For an oscillator of unit mass and natural frequency ω, this function is

$$\phi_n(x) = (2^n n!)^{-1/2} (\omega/\pi\hbar)^{1/4}\, e^{-\omega x^2/(2\hbar)}\, H_n\big(x\sqrt{\omega/\hbar}\big). \tag{3}$$

The positional probability density in the ground state $|0\rangle$ is the normalized Gaussian

$$|\phi_0(x)|^2 = (\omega/\pi\hbar)^{1/2}\, e^{-\omega x^2/\hbar}. \tag{4}$$

Recall that $|0\rangle$ is a *minimum uncertainty* state, in which the product of the standard deviations in $\hat{x}$ and $\hat{p}$ is given by $\Delta\hat{x}\,\Delta\hat{p}$ is exactly $\frac{1}{2}\hbar$. For all the excited states, $\Delta\hat{x}\,\Delta\hat{p} > \frac{1}{2}\hbar$.

The stationary states $|n\rangle$ of the linear harmonic oscillator are given by the product of a Gaussian factor and the Hermite polynomial of order *n*.

The radiation problem we are considering, however, is only formally equivalent to a quantum mechanical harmonic oscillator. What significance do 'position' and 'momentum' have in this case? They are just the *hermitian* operators from which the *nonhermitian* annihilation and creation operators are constructed as linear

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combinations – in the same way that a complex number $z = x + iy$ and its complex conjugate $z^* = x - iy$ are linear combinations of the two real numbers x and y. Thus, we *define* the hermitian operators $\hat{x}$ and $\hat{p}$ as the linear combinations

$$\hat{x} = (2\hbar/\omega)^{1/2}\,\frac{(\hat{a}+\hat{a}^\dagger)}{2} \quad \text{and} \quad \hat{p} = (2\hbar\omega)^{1/2}\,\frac{(\hat{a}-\hat{a}^\dagger)}{2i}\,. \tag{5}$$

The inverse relations corresponding to (5) are

$$\hat{a} = \hat{x}\,(\omega/2\hbar)^{1/2} + i\hat{p}\,(2\hbar\omega)^{-1/2} \quad \text{and}$$
$$\hat{a}^\dagger = \hat{x}\,(\omega/2\hbar)^{1/2} - i\hat{p}\,(2\hbar\omega)^{-1/2}\,. \tag{6}$$

The basic commutation relation $[\hat{a}\,,\,\hat{a}^\dagger] = \hat{I}$ then goes over into the standard position-momentum commutation relation $[\hat{x}\,,\,\hat{p}] = i\hbar\hat{I}$.

• The important point is that, although the nonhermitian operators $\hat{a}$ and $\hat{a}^\dagger$ are not directly measurable, the hermitian operators $\hat{x}$ and $\hat{p}$ *are*, in principle, measureable.

We have made them look exactly like the position and momentum operators of a linear harmonic oscillator, but there is no actual mechanical oscillator here. On the other hand, they *are* observables, which can be determined by suitable measurements on the electromagnetic fields associated with the radiation. They are called *quadratures* in the quantum optics context. We shall therefore speak of the $\hat{x}$-quadrature and $\hat{p}$-quadrature from now on[2]. The Hamiltonian of the radiation field, too, now looks like that of a harmonic oscillator of unit mass, with the ground state energy subtracted out (so that the zero-photon state has zero energy):

$$\hat{H} = \hbar\omega\,\hat{a}^\dagger\,\hat{a} = \tfrac{1}{2}\,\hat{p}^2 + \tfrac{1}{2}\,\omega^2\hat{x}^2 - \tfrac{1}{2}\,\hbar\omega\,\hat{I}. \tag{7}$$

We can now represent the various states of the radiation field in terms of x-space wave functions, if we so desire. We could also use the p-space wave functions, or any

[2] We reiterate that you must *not* regard $\hat{x}$ and $\hat{p}$ as the 'position' and 'momentum' of a photon!

other representation, of course. But the wave functions in x-space are more familiar ones, and suffice for our purposes.

• The time evolution of any initial, prepared, state $|\Psi(0)\rangle$ of the radiation field as it propagates in free space is given by the time evolution of the corresponding initial wave function $\psi(x, 0)$ as governed by the oscillator Hamiltonian (7).

• That is, we need to study only the quantum mechanical evolution of any initial wave function in the parabolic potential $\frac{1}{2}\omega^2 x^2$.

• Moreover, we can use Ehrenfest's theorem to study the evolution of the expectation values of the $\hat{x}$ and $\hat{p}$ quadratures and their higher moments.

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3.1 *Wave Functions in x-Space*

Having outlined the strategy, it remains to specify the wave functions for the initial states of interest to us. The n-photon Fock state $|n\rangle$, of course, has precisely the x-space wave function $\phi_n(x)$ given by (3). What about the CS $|\alpha\rangle$? Let us use the shorthand notation $\alpha(x)$ to denote the corresponding wave function $\psi_\alpha(x) \equiv \langle x|\alpha\rangle$. We can compute it from the definition of the CS, given in (2), after inserting the known expression for $\langle x|n\rangle = \phi_n(x)$ in the sum over n. A formula for the generating function of the Hermite polynomials is needed to carry out the summation in the resulting expression. But there is a simpler and more instructive method. From the eigenvalue equation $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$ it follows that

$$\langle x|(\hat{a} - \alpha)|\alpha\rangle = 0. \quad (8)$$

Now, as (6) shows, $\hat{a}$ is a linear combination of $\hat{x}$ and $\hat{p}$. In the x-representation, as we learn in elementary courses on QM, the action of $\hat{x}$ merely corresponds to multiplication by x, while the action of $\hat{p}$ is that of the derivative operator $-i\hbar \mathrm{d}/\mathrm{d}x$. Therefore (8) becomes, *Every* member of the whole family of coherent states has a Gaussian probability density, and is a minimum uncertainty state

when written out in the x-representation,

$$\left[(\hbar/2\omega)^{1/2}\mathrm{d}/\mathrm{d}x + (\omega/2\hbar)^{1/2}x - \alpha\right]\alpha(x) = 0. \qquad (9)$$

(Remember that the term α inside the square brackets is a complex constant, while $\alpha(x)$ is the x-space wave function corresponding to the state $|\alpha\rangle$.) This ordinary, first-order differential equation is easy to solve. There is no additive constant of integration because a normalizable wave function must vanish at $x = \pm\infty$. The solution is a Gaussian multiplied by an x-dependent phase factor whose modulus is unity. Thus $\alpha(x)$ is a Gaussian modulated by an oscillatory factor. (We leave it to the reader to find $\alpha(x)$ explicitly.) The corresponding probability density is a normalized Gaussian, given by

$$|\alpha(x)|^2 = (\omega/\pi\hbar)^{1/2}\,\exp\,\left[-(\omega/\hbar)\left(x - \alpha_1\sqrt{2\hbar/\omega}\right)^2\right], \qquad (10)$$

where α_1 is the real part of α. We conclude that:

- *Every* member of the whole family of coherent states has a Gaussian probability density, and is a minimum uncertainty state, with $\Delta\hat{x}\,\Delta\hat{p} = \frac{1}{2}\,\hbar$.

Compare (10) with (4) for the ground state of the oscillator. The Gaussian in (10) is *displaced*, with its peak at $x = (2\hbar/\omega)^{1/2}\,\alpha_1$ rather than $x = 0$. In the same way, we can show that the p-space probability density corresponding to the CS $|\alpha\rangle$ is also a Gaussian, with its maximum at $p = (2\hbar\omega)^{1/2}\,\alpha_2$, where α_2 is the imaginary part of α.

The real and imaginary parts of the complex parameter α therefore determine the extent to which the peaks of the probability densities of the CS $|\alpha\rangle$ are *displaced* with respect to that of the zero-photon state $|0\rangle$.

- The real and imaginary parts of the complex parameter α therefore determine the extent to which the peaks of the probability densities of the CS $|\alpha\rangle$ in x-space and p-space, respectively, are *displaced* with respect to that of the zero-photon state $|0\rangle$. This is why the CS $|\alpha\rangle$ is also called a displaced vacuum state.

The expectation values of $\hat{x}$ and $\hat{p}$ are obviously given by the locations of the centres of the respective Gaussian probability densities, namely, $\langle\hat{x}(0)\rangle = (2\hbar/\omega)^{1/2}\,\alpha_1$

and $\langle\hat{p}(0)\rangle = (2\hbar\omega)^{1/2}\alpha_2$. We have indicated the time argument in the expectation values, in anticipation of the fact that these quantities will change with time, as we shall see shortly in Section 4.

Next, we turn to the x-space wave function $\langle x|\alpha, m\rangle$ corresponding to the PACS $|\alpha, m\rangle$. This wave function is not as simple as a Gaussian modulated by a phase factor. Apart from a normalization factor, it is given by a differential operator acting on $\alpha(x)$, because

$$\langle x|\alpha, m\rangle \propto \langle x|(\hat{a}^\dagger)^m|\alpha\rangle$$
$$= \left[-(\hbar/2\omega)^{1/2}\mathrm{d}/\mathrm{d}x + (\omega/2\hbar)^{1/2}x\right]^m \alpha(x). \quad (11)$$

The result can be quite a complicated function, especially for large values of m. The corresponding probability density of $\langle x|\alpha, m\rangle$ will no longer be unimodal, i.e., it will no longer have a single maximum. With increasing m, the number of maxima and minima that it has also increases. We will illustrate the specific case $m = 1$ graphically in Section 4. This feature has interesting consequences which show up in the expectation values of $\hat{x}$ and $\hat{p}$ and in their higher moments, and also in the dynamics of the state as it evolves, *even in a simple parabolic potential*. These aspects help us understand why the CS is referred to as a 'classical' state, while the PACS is not.

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4. Dynamics and Ehrenfest's Theorem

We are ready, now, to consider the time evolution of an initial coherent state of radiation as it propagates in free space. That is, we look at the state $|\Psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\Psi(0)\rangle$ where $\hat{H} = \hbar\omega\,\hat{a}^\dagger\,\hat{a}$, for the initial state $|\Psi(0)\rangle = |\alpha\rangle$. Entirely equivalently, as we have explained in Section 3, we could examine the evolution (according to the Schrödinger equation) of the wave function of a particle of unit mass in the oscillator potential $\frac{1}{2}\omega^2 x^2$, given the initial wave function $\psi(x, 0) = \alpha(x)$. This latter procedure is the one we shall adopt.

A *classical* particle with a given total energy *E* when placed in the parabolic potential $\frac{1}{2}\omega^2 x^2$ would oscillate between the classical *turning points* $x = \pm\sqrt{2E}/\omega$, where the potential energy becomes equal to *E*.

Note that, for any time-independent Hamiltonian, we have the general result $\langle\Psi(t)|\hat{H}|\Psi(t)\rangle = \langle\Psi(0)|\hat{H}|\Psi(0)\rangle$, since $\hat{H}$ commutes with itself. This is true no matter what the initial state is. Therefore, even though $|\alpha\rangle$ is not an eigenstate of $\hat{H} = \hbar\omega\, \hat{a}^\dagger\, \hat{a}$, the mean photon number and the average energy remain equal to $|\alpha|^2$ and $\hbar\omega\,|\alpha|^2$, respectively, at all times. The expectation value of the Hamiltonian of the equivalent particle is therefore $E = \hbar\omega\,|\alpha|^2$, and this remains constant under time evolution.

Now consider what a *classical* particle with a given total energy E would do when placed in the parabolic potential $\frac{1}{2}\omega^2 x^2$. It would oscillate between the classical *turning points* $x = \pm\sqrt{2E}/\omega$, where the potential energy becomes equal to E. The particle cannot move beyond these turning points. Quantum mechanically, however, a particle with some given expectation value $\langle\hat{H}\rangle = E$ has, *in general*, a wave that extends over the full range $-\infty < x < \infty$. For instance, each of the Fock state eigenfunctions $\phi_n(x) = \langle x|n\rangle$ in (3) extends over the full range in x. So does the wave function $\alpha(x)$ of the CS $|\alpha\rangle$.

How do these wave functions change with time? In the stationary state $|n\rangle$, of course, the initial wave function $\phi_n(x)$ merely acquires a multiplicative phase factor and becomes $e^{-iE_n t/\hbar}\,\phi_n(x) = e^{-in\omega t}\,\phi_n(x)$ at any time t. All expectation values in this state remain unchanged in time. In particular, $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$ remain equal to zero for all t in each Fock state. But something quite remarkable happens to an initial CS $|\alpha\rangle$. We have

$$\begin{aligned} |\alpha\rangle \rightarrow e^{-i\hat{H}t/\hbar}|\alpha\rangle &= e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n\, e^{-ni\omega t}}{\sqrt{n!}}\,|n\rangle \\ &= |\alpha\, e^{-i\omega t}\rangle. \end{aligned} \tag{12}$$

In other words:

• An initial coherent state remains a coherent state at all times.

• The complex parameter labelling the state retains its magnitude, while its real and imaginary parts oscillate sinusoidally with a frequency ω.

The wave functions in x-space and p-space remain Gaussians, modulated by pure phase factors. The probability density retains its Gaussian form both in x and in p. But the centre of each Gaussian, which is also the expectation value of the corresponding quadrature, now oscillates in time with a frequency ω, according to

$$\left.\begin{aligned} \langle \hat{x}(t)\rangle &= (2\hbar/\omega)^{1/2}\,(\alpha_1 \cos \omega t + \alpha_2 \sin \omega t), \\ \langle \hat{p}(t)\rangle &= (2\hbar\omega)^{1/2}\,(\alpha_2 \cos \omega t - \alpha_1 \sin \omega t). \end{aligned}\right\} \quad (13)$$

In other words, these expectation values behave exactly like the position and momentum of a *classical* linear harmonic oscillator. The correspondence is even more exact: The amplitude of oscillation in x-space is easily seen to be $(2\hbar/\omega)^{1/2}\,|\alpha|$. But since the mean energy in the state is $E = \hbar\omega\,|\alpha|^2$, this is *precisely* the position of the classical turning point $\sqrt{2E}/\omega$.

• The quantum mechanical expectation values $\langle \hat{x}(t)\rangle$ and $\langle \hat{p}(t)\rangle$ in the CS $|\alpha\rangle$ behave *exactly* like the classical position and momentum of a linear harmonic oscillator.

• The state remains a minimum uncertainty state at all times.

These features support the use of the term 'classical' state of radiation to describe the coherent state $|\alpha\rangle$.

Figures 1a-c depict these conclusions graphically. (For illustrative purposes, we have set $\hbar = 1$ and used the values $\omega = 0.005$ and $\alpha = 1$ in all the figures in this article.) The parabola in black shows the oscillator potential, and the violet bell-shaped Gaussian depicts the

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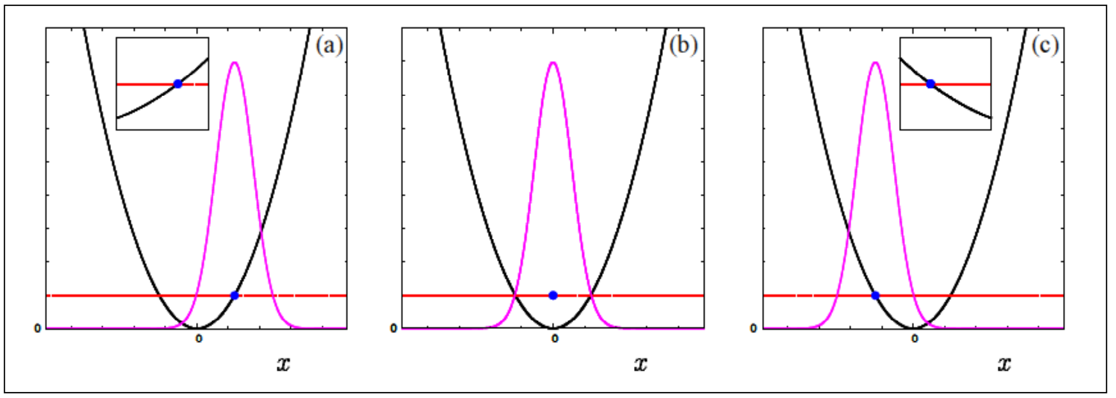

Figure 1. Periodic motion of the positional probability density (violet curve) and $\langle \hat{x}(t) \rangle$ (blue dot) for a coherent state. (a) $t = 0$; (b) $t = \pi/(2\omega)$;(c) $t = \pi/\omega$. The parabolic curve in black is the oscillator potential. The horizontal red dotted line indicates the mean energy of the state. The probability density profile does not change shape during the time evolution, but merely translates back and forth.

probability density corresponding to the CS at the instants (a) $t = 0$, $(b)\,\pi/2\omega$, and (c) π/ω, respectively. (This covers one half of a complete oscillation.) The horizontal red dotted line corresponds to the value of the mean energy E in the CS, and it intersects the potential at the two classical turning points. The blue dot on this line represents the expectation value $\langle \hat{x}(t) \rangle$. Its location on this line illustrates how it oscillates back and forth like a classical oscillator would. As we have shown above, the maximum and minimum values of $\langle \hat{x}(t) \rangle$ coincide with the classical turning points. This is indicated more clearly in the insets in *Figures* 1a and c.

The dynamics of the higher powers of $\hat{x}$, $\hat{p}$ and their combinations can be established similarly, with the help of the Ehrenfest theorem.

Let us now turn to the case when the initial state is a PACS. As we have seen in Section 3, the probability density (either in x-space or in p-space) is not a Gaussian for a PACS. The state is not a minimum uncertainty state at any time. Moreover, this density does not retain its shape as it evolves under the influence of the oscillator potential. In *Figures* 2a–c, we show what happens in the case of an initial one-photon-added coherent state $|\alpha, 1\rangle$. The colour coding is exactly the same as in *Figure* 1, as are the numerical values of the constants. The dotted red line again corresponds to the mean energy in the PACS, $E = \hbar\omega\,\langle \hat{a}^\dagger\,\hat{a} \rangle$. (But remember that $\langle \hat{a}^\dagger\,\hat{a} \rangle$ is no

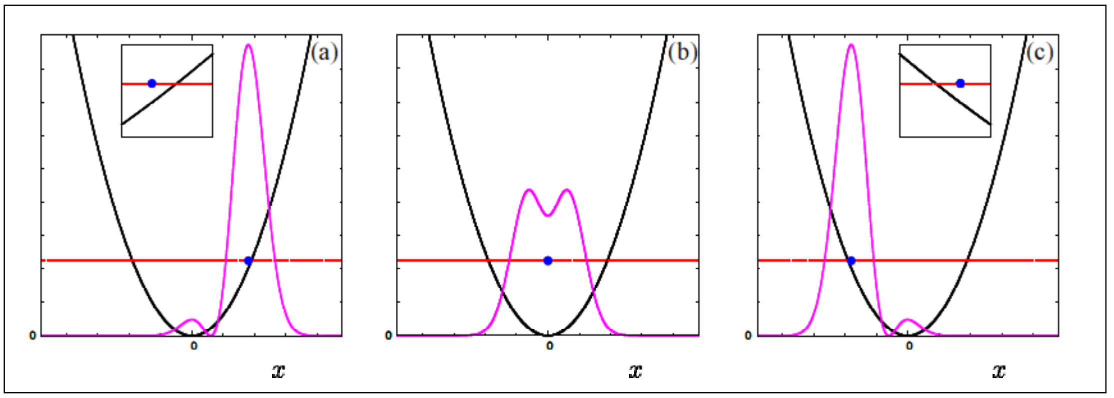

Figure 2. Periodic motion of the positional probability density (violet curve) and $\langle\hat{x}(t)\rangle$ (blue dot) for a 1-photon-added coherent state. (a) $t = 0$; (b) $t = \pi/(2\omega)$; (c) $t = \pi/\omega$. The horizontal red dotted line indicates the mean energy E of the state. Note that (i) the probability density profile has two maxima, (ii) it changes shape during the time evolution, and (iii) $\langle\hat{x}(t)\rangle$ does not reach the classical turning points at any time.

longer equal to $|\alpha|^2$, its value in the case of a CS.) We see that the probability density in x-space has two maxima rather than just one. Moreover, the curve changes shape in a rather drastic (but regular) fashion as it oscillates back and forth.

The expectation values $\langle\hat{x}(t)\rangle$ and $\langle\hat{p}(t)\rangle$ continue to vary sinusoidally, as in simple harmonic motion. This feature is guaranteed by the Ehrenfest theorem, and is implicit in equation (9) appearing in Part 1 of this two-part article, namely, $\mathrm{d}\langle\hat{x}\rangle/\mathrm{d}t = \langle\hat{p}\rangle$ and $\mathrm{d}\langle\hat{p}\rangle/\mathrm{d}t = -\omega^2\,\langle\hat{x}\rangle$. These equations remain valid for all states and all times, since the Hamiltonian is quadratic in $\hat{x}$ and $\hat{p}$. There is, however, an important difference between the present case and that of a CS: *the amplitude of oscillation falls short of the classical turning point* $\sqrt{2E}/\omega$ corresponding to the total energy E. This is shown more clearly in the insets in *Figures* 2a and c. This feature, and the fact that the state is never a minimum uncertainty state, enable us to say that we are now dealing with a nonclassical state of radiation.

There is a simple way of understanding why the amplitude of oscillation of $\langle\hat{x}(t)\rangle$ falls below the classical turning point. Recall that $\langle\hat{x}(t)\rangle$ remains equal to zero for the Fock state $|m\rangle$, while it periodically reaches the classical turning point for the coherent state $|\alpha\rangle$. The PACS $|\alpha, m\rangle$ interpolates between the Fock state and

The values of $\Delta\hat{x}$ and $\Delta\hat{p}$ in any CS $|\alpha\rangle$ are precisely those in the vacuum state. (Hence a CS is a minimum uncertainty state, but not a squeezed state.)

the CS. It is therefore plausible that the amplitude of oscillation of $\langle\hat{x}(t)\rangle$ for the PACS lies in between 0 and $\sqrt{2E}/\omega$.

Finally, let us consider another class of states of the radiation field that are also nonclassical: *squeezed states.* As explained in *Box* 1, a state of light is said to be squeezed in the $\hat{x}$-quadrature if the uncertainty $\Delta\hat{x}$ in that state falls below $\sqrt{\hbar/(2\omega)}$, which is its value in the vacuum or zero-photon state. Similarly, it is squeezed in the $\hat{p}$-quadrature if $\Delta\hat{p} < \sqrt{\hbar\omega/2}$, its value in the vacuum state. The values of $\Delta\hat{x}$ and $\Delta\hat{p}$ in any CS $|\alpha\rangle$ are precisely those in the vacuum state. (Hence a CS is a minimum uncertainty state, but not a squeezed state.)

Box 1. Squeezed States

The commutation relation $[\hat{x}\,,\,\hat{p}] = i\hbar\,\hat{I}$ for the conjugate quadratures $\hat{x}$ and $\hat{p}$ pertaining to a quantum mechanical system implies that the product of their uncertainties (i.e., their standard deviations) satisfies the Heisenberg Uncertainty Principle $\Delta\hat{x}\,\Delta\hat{p} \geq \frac{1}{2}\hbar$ in any state of the system. It follows from this uncertainty relation that $\hat{x}$ and $\hat{p}$ cannot both be specified simultaneously to arbitrary accuracy in any state of the system. In particular, if one of them is known exactly, the other cannot be fixed at all. This is an *inherent* feature of quantum physics: for such a relation between standard deviations does not exist in classical physics, where the simultaneous specification of conjugate observables to arbitrary accuracy is possible if the measuring device has sufficiently high resolution.

The coherent state $|\alpha\rangle$ is a minimum uncertainty state because $\Delta\hat{x}\,\Delta\hat{p}$ is equal to its least possible value, $\frac{1}{2}\hbar$, in that state. In this sense it is the nearest we can get to a classical state. The individual uncertainties $\Delta\hat{x}$ and $\Delta\hat{p}$ in the CS $|\alpha\rangle$ are given by $\Delta\hat{x} = \sqrt{\hbar/(2\omega)}$ and $\Delta\hat{p} = \sqrt{\hbar\omega/2}$, respectively. Note that these values are independent of the parameter α. They are therefore equally valid for the vacuum state $|0\rangle$, which corresponds, of course, to $\alpha = 0$.

A *squeezed* state is one in which either $\Delta\hat{x}$ or $\Delta\hat{p}$ drops below its value in a CS (or in the vacuum state). The uncertainty in the other conjugate observable will, of course, be greater than that in a CS, because of the uncertainty principle. Squeezed light can be produced with specific superpositions of the photon number states. The *squeezed vacuum state* is obtained by applying the operator $\exp(\beta\,\hat{a}^{\dagger 2} - \beta^*\,\hat{a}^2)$, where β is any complex number, to $|0\rangle$. The squeezed state we consider in the text is given by the superposition $\frac{1}{2}\big(\sqrt{3}\,|0\rangle + |1\rangle\big)$. This state is squeezed in the $\hat{x}$-quadrature. While it is not very feasible to achieve this state of radiation experimentally, it is easy to calculate various quantities explicitly in it. This is why we have chosen it for illustrative purposes.

Now consider, at $t = 0$, the normalized state given by the following superposition of the zero-photon and one-photon Fock states:

$$|\chi(0)\rangle = \tfrac{1}{2}\big(\sqrt{3}\,|0\rangle + |1\rangle\big). \tag{14}$$

The mean energy in the state is $E = \frac{1}{4}\,\hbar\omega$. We know that $|0\rangle$ is a minimum uncertainty state, with $\Delta\hat{x} = \sqrt{\hbar/(2\omega)}$, while in the state $|1\rangle$ we have $\Delta\hat{x} = \sqrt{3\hbar/(2\omega)}$. When superposed as in (14), however, the resultant state gets squeezed in the $\hat{x}$-quadrature! It is easy to show that, in the state $|\chi(0)\rangle$,

$$\Delta\hat{x}(0) = \sqrt{3\hbar/(8\omega)} \quad \text{and} \quad \Delta\hat{p}(0) = \sqrt{3\hbar\omega}/2. \tag{15}$$

Since $\frac{3}{8} < \frac{1}{2}$, the state is squeezed in the $\hat{x}$-quadrature. Now let the state evolve under the Hamiltonian $\hat{H} = \hbar\omega\,\hat{a}^\dagger\,\hat{a}$, corresponding to the propagation of the radiation in free space. It follows readily that

$$|\chi(t)\rangle = \tfrac{1}{2}\big(\sqrt{3}\;|0\rangle + e^{-i\omega t}\;|1\rangle\big). \tag{16}$$

A short calculation gives the expressions

$$\begin{aligned}\Delta\hat{x}(t) &= [(3\hbar/8\omega)(2 - \cos^2\omega t)]^{1/2} \quad \text{and}\\ \Delta\hat{p}(t) &= [(3\hbar\omega/8)(2 - \sin^2\omega t)]^{1/2}.\end{aligned} \tag{17}$$

The $\hat{x}$-quadrature is squeezed whenever $\cos^2\omega t > \frac{2}{3}$. In particular, it is squeezed at $t = 0$ and $t = \pi/\omega$, i.e., at the end-points of the oscillation of $\langle\hat{x}(t)\rangle$. The $\hat{p}$-quadrature is squeezed whenever $\sin^2\omega t > \frac{2}{3}$. In particular, it is squeezed at $t = \pi/(2\omega)$ and $3\pi/(2\omega)$, i.e., at the mid-point of the oscillation. The uncertainty product $\Delta\hat{x}(t)\,\Delta\hat{p}(t)$ itself remains always above the minimum uncertainty value $\frac{1}{2}\hbar$, varying periodically from a minimum value of $3\sqrt{8}\,\hbar/16$ to a maximum value of $9\hbar/16$, with a time period π/ω. The x-space probability density works out to

$$\begin{aligned}|\langle x|\chi(t)\rangle|^2 &= (\omega/\pi\hbar)^{1/2}\,e^{-\omega x^2/\hbar}\,\times\\ &\left[\tfrac{3}{4} + (3\omega/2\hbar)^{1/2}\,x\,\cos\,\omega + \omega x^2/(2\hbar)\right].\end{aligned} \tag{18}$$

The uncertainty product $\Delta\hat{x}(t)\,\Delta\hat{p}(t)$ of the squeezed state remains always above the minimum uncertainty value $\frac{1}{2}\hbar$.

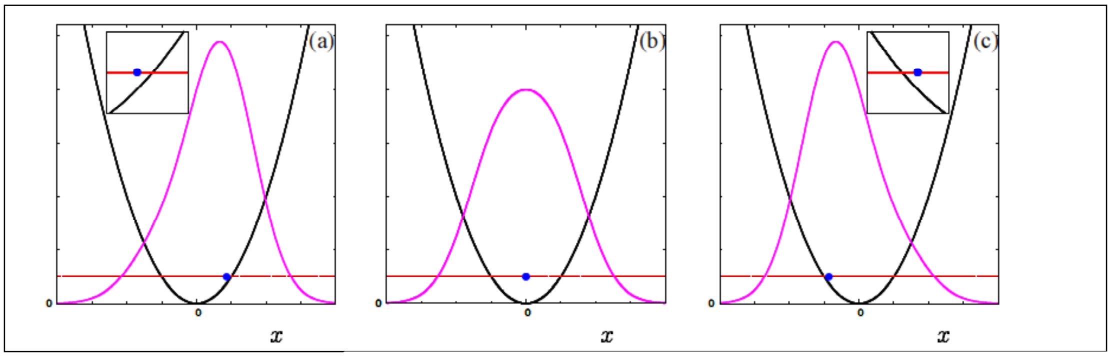

Figure 3. Periodic motion of the positional probability density (violet curve) and $\langle \hat{x}(t) \rangle$ (blue dot) for a squee-zed state. (a) t = 0; (b) $t = \pi/(2\omega)$; (c) $t = \pi/\omega$. Note that the probability density profile is unimodal,but changes shape during the time evolution. As in the case of the PACS, $\langle \hat{x}(t) \rangle$ does points at any time.

The expectation value $\langle \hat{x} \rangle$ is identically equal to zero in any Fock state $|n\rangle$. However, when $|0\rangle$ and $|1\rangle$ are superposed as in $|\chi(0)\rangle$ and allowed to evolve in time, the time-dependent coefficient in $|\chi(t)\rangle$ shows the powerful nature of quantum mechanical superposition! $\langle \hat{x}(t) \rangle$ undergoes simple harmonic motion, according to $\langle \hat{x}(t) \rangle = (3\hbar/8\omega)^{1/2} \cos \omega t$. *Figures* 3a–c show, as in the earlier cases, the periodic motion of the expectation value $\langle \hat{x}(t) \rangle$ in the oscillator potential. Once again, the probability density in x-space changes its shape periodically, although it remains unimodal at all times. As in the case of the PACS, $\langle \hat{x}(t) \rangle$ does not reach the classical turning points (located at $\pm\sqrt{\hbar/(2\omega)}$ in this case). This is, ofcourse, the behaviour we would expect of a nonclassical state of radiation.

Our treatment of a few aspects of some nonclassical states of light has been essentially pedagogical. It must be mentioned that there are more stringent quantitative measures of the nonclassicality of quantum mechanical states in general. Even among states of radiation, there are many other interesting nonclassical states: for instance, the so-called 'cat' states such as the even and odd coherent states $|\alpha\rangle \pm |-\alpha\rangle$ and the Yurke–Stoler state $|\alpha\rangle + i\,|-\alpha\rangle$; the squeezed vacuum state; and so on. In all these cases, the Ehrenfest theorem provides a convenient way to analyse the dynamics, and to understand (via the behaviour of the higher moments of the quadratures concerned) the role played by quantum fluctuations.

We have restricted ourselves to the simplest case of propagation of radiation in free space. As we have seen, this corresponds to effective motion in a quadratic or harmonic oscillator potential in the quadrature $\hat{x}$. In contrast, propagation in *nonlinear media* is of great interest in quantum optics. In *Box* 2, we have commented briefly on a case of considerable practical importance, the *Kerr medium.* The equivalent problem in terms of $\hat{x}$ and $\hat{p}$ now involves nonquadratic Hamiltonians. As explained in Part 1, Ehrenfest's theorem continues to provide, under suitable conditions, a good approximation to the dynamics in such cases as well.

Box 2. Wave Packet Revivals in Nonlinear Media

The term *Kerr medium* is used for a specific kind of nonlinear optical medium that displays an intensity-dependent refractive index and several interesting associated phenomena. Certain dyes and semiconductor materials are good Kerr-like media. In the text, we have seen how propagation of radiation in a vacuum is equivalent to time evolution governed by the quadratic Hamiltonian $\hbar\omega\, \hat{a}^\dagger\, \hat{a}$. Propagation through a Kerr medium is modelled by adding to this a quartic term proportional to $\hat{a}^{\dagger 2}\, \hat{a}^2$. In terms of the hermitian operators $\hat{x}$ and $\hat{p}$, this will immediately imply a Hamiltonian that is higher than quadratic order in those quadratures. It follows from the commutation relation $[\hat{a}\,,\, \hat{a}^\dagger] = \hat{I}$, however, that $\hat{a}^{\dagger 2}\, \hat{a}^2 = (\hat{a}^\dagger\, \hat{a})^2 - \hat{a}^\dagger\, \hat{a}$. The Kerr Hamiltonian is therefore completely expressible as a function of the photon number operator. Hence its eigenstates are again those of $\hat{a}^\dagger\, \hat{a}$. In particular, their x-space wave functions continue to be given by $\phi_n(x)$ as in 3. This fact is of help in calculations.

We have also seen how a coherent state is related to a Gaussian wave form. When propagating through a Kerr medium, light in a CS will almost immediately lose its Gaussian property and spread out into different wave forms. Remarkably enough, however, the wave packet could regain its Gaussian shape periodically, at specific instants of time. The explanation of this *revival* phenomenon lies in the quantum interference between the basis states $|n\rangle$ comprising the state of the radiation field. Further, at specific instants between two succesive revivals, the wave packet could reconstitute itself into two, three, $\ldots$ Gaussian wave packets of smaller amplitude. These are referred to as *fractional revivals* of the original state. The occurence of revivals and fractional revivals depends crucially on the higher-order term $\hat{a}^{\dagger 2}\, \hat{a}^2$ in the effective Hamiltonian of the system. A large class of quantum states of radiation, including the PACS and certain squeezed states, could display revivals and fractional revivals when propagating through a Kerr medium. (See *Box* 3.)

Box 3.

The Wigner function is an important measure of the extent of non-classicality of a quantum mechanical state. It is not a probability distribution in the conventional, classical sense of the term, as it need not be strictly non-negative for all values of its argument. The three-dimensional figure (*Figure* A) shows the Wigner function for an even coherent state, plotted against the real and imaginary parts of its complex argument. The basal plane is the zero level, while positive (respectively, negative) values lie above (respectively, below) it. In contrast, the Wigner function for a pure coherent state does not dip below the zero level, affirming its 'classicality'.

When an initial even coherent state propagates through a nonlinear optical medium such as a Kerr medium (modelled by the Hamiltonian mentioned in *Box* 2), it undergoes periodical revivals. The set of figures (*Figure* B) shows the projection onto the basal plane of the Wigner function for such an initial state through a full revival period, captured at specific fractions of this period. The colour coding alongside indicates the extent to which the function rises above (or falls below) the zero level.

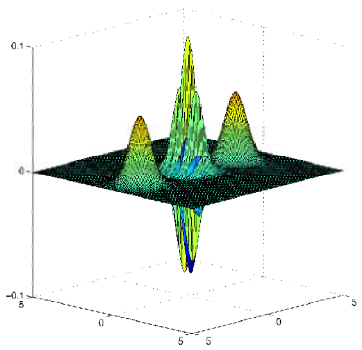

Figure A.

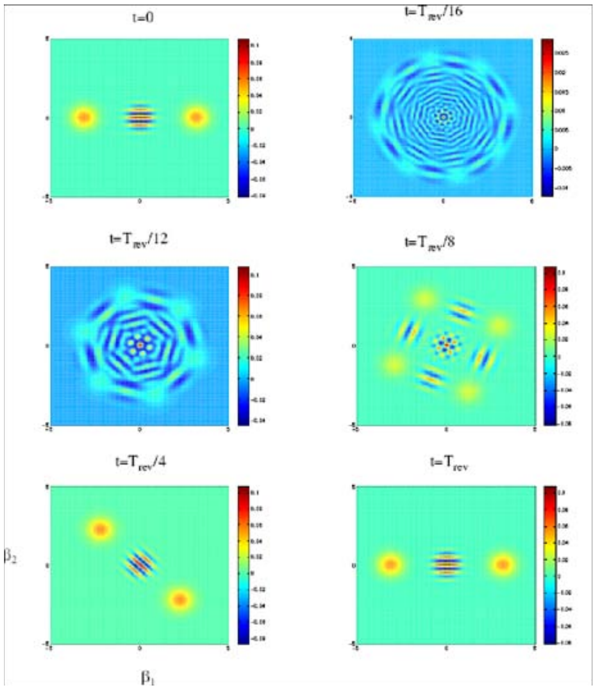

Figure B.

A student exposed for the first time to quantum physics is faced with a collection of radical ideas such as the uncertainty principle, the inherently probabilistic nature of quantum physics, light as quanta of energy, tunnelling of particles through potential barriers, quantum mechanical superposition of states, quantum states with no classical analogues, and so on. All of these seem to defy 'physical intuition' as we know it in the macroscopic 'world of middle dimensions' which we perceive directly and experience. In this situation, the Ehrenfest theorem stands out as the most important link between the familiar classical world and the unfamiliar terrain of the quantum world. The physicist responsible for establishing this connection was also an expositor of the highest clarity and a teacher of remarkable excellence. As the *Wikipedia* article on Paul Ehrenfest says, quoting the words of Albert Einstein himself: "He was not merely the best teacher in our profession whom I have ever known; he was also passionately preoccupied with the development and destiny of men, especially his students ··· to encourage youthful talent – all this was his real element ··· ."

The Ehrenfest theorem stands out as the most important link between the familiar classical world and the unfamiliar terrain of the quantum world.

Suggested Reading

[1] J J Sakurai, *Modern Quantum Mechanics*, International Student Edition, Addison Wesley Longman, Reading, Massachusetts, 1999.

[2] C Gerry and P Knight, *Introductory Quantum Optics*, Cambridge University Press, Cambridge, 2005.

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