Dawn of Science

20. Calculus is Developed in Kerala

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Foundations of calculus were developed by a school of mathematicians in Kerala during 1400–1600, years before similar developments took place in the west.

An ancient text from Kerala has a verse, giving the circumference of a circle, the translation of which goes as:

"*Multiply the diameter by four. Subtract from it and add to it alternatively the quotients obtained by dividing four times the diameter to the odd numbers 3,5 etc.* "

In modern notation this reduces to a remarkable infinite series expansion for $\pi/4$ given by

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots . \qquad (1)$$

This result – usually attributed to Gregory (1638–1675) and others who were to be born more than a century after the text in question was authored – is one of the many gems to be found in the ancient Kerala texts (especially the one called *Yuktibhasha*) which laid the foundation for a branch of mathematics that we now call calculus.

Previous parts:
Resonance, Vol.15: p.498, p.590, p.684, p.774, p.870, p.1009, p.1062; Vol.16: p.6, p.110, p.274, p.304, p.446, p.582, p.663, p.770, p.854, p.950, p.103; Vol.17: p.6.

The discovery of the contributions from the Kerala school of mathematicians is comparatively recent and many details are still being probed extensively. This is inspite of the fact that some of these ancient Kerala texts have been referred to explicitly in an article by C M Whish who – having learnt Malayalam and collected palm leaf manuscripts from Kerala – found to his astonishment a "complete system of fluxions" in them. He published a paper in 1834 in the *Transactions of the Royal Asiatic*

Keywords
Yuktibhasha, Madhava, tantrasamgrha, Kerala calculus.

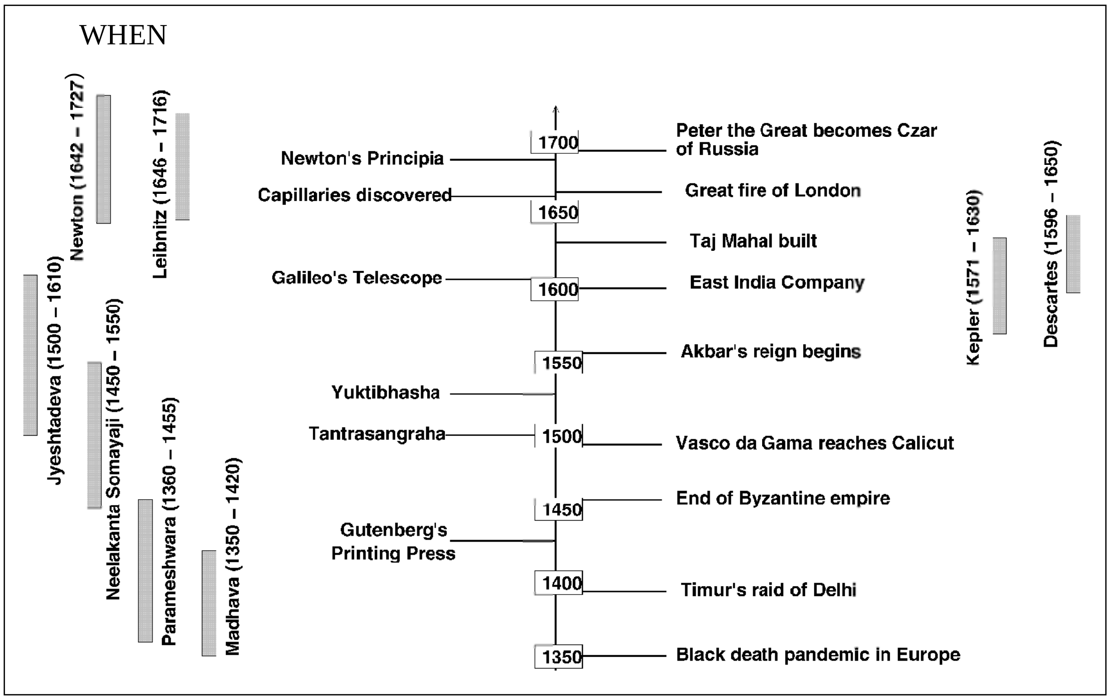
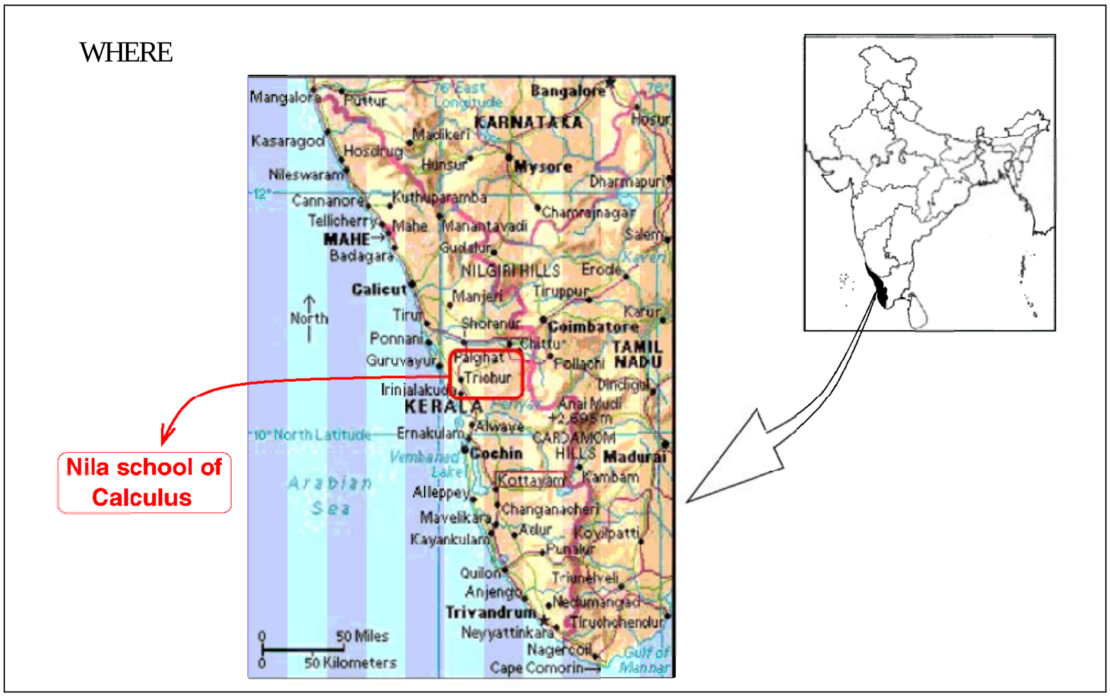


Figure 1.

Figure 2.



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Society of Great Britian and Ireland with a fairly explicit title 'On the Hindu quadrature of the circle and the infinite series of the proportion of the circumference to the diameter exhibited in the four *sastras*, the *Tantrasangraham, Yukti-Bhasha, Carana Padhati* and *Sadratnamala*'.

In addition to the result quoted above, these texts contain the infinite series expansion for $\tan^{-1}(x)$ (from which the above result is but one step; you just put $x = 1$), the series expansion for sin and cos, and the indefinite integral of x^n – just to list a few. Virtually all these results were obtained by using techniques which can be thought of as belonging to calculus.

Who are these mathematicians and how did they develop these techniques? Though many of the details are still somewhat sketchy, historians of Indian mathematics have now put together the following picture. Most of these developments took place in villages around a river called Nila in the ancient days (and currently called river Bharatha, the second longest river in Kerala) during 1300–1600 or so. One of the key villages was called Sangama-grama in the ancient times and is thought of as referring to the village Irinhalakkuta (about 50 km to the south of Nila) in present day Kerala. (There are a few other candidates, like Kudalur and Tirunavaya, for which one could raise arguments in favour of them being the Sangama-grama; so this issue is not completely settled.) What is more certain was the existence of a remarkable lineage of mathematicians in Sangama-grama of which Madhava ($\sim$ 1350–1420) seems to be the one who discovered many of the basic ideas of calculus. The infinite series for sine, cosine, arctan as well as rudiments of integration are attributed to Madhava by many later sources. He was strongly influenced, like many other Indian scholars of that period, by Aryabhata – which is no surprise since *Aryabhateeyam* (circa 500 AD) was a very influential text in India as well as (through its

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translations) in the Arab world and medieval Europe.

Madhava founded a lineage which lasted possibly till the late seventeenth century. His student Parameshwara ($\sim$ 1360–1455) was an astronomer-mathematician who had authored more than two dozen works and made astronomical observations for a very extended period of nearly five decades. It appears that his son Dhamodara was the teacher of another key figure, Neelakanta Somayaji ($\sim$ 1450–1550) who is the author of *Tantrasangraha* which contains extensive discussions on astronomy and related mathematics. Another student of Dhamodara was Jyeshtadeva ($\sim$ 1500–1610), the author of *Yuktibhasha* which probably has the clearest exposition of some of the topics in calculus.

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The quotation in the beginning of this article is quoted in *Yuktibhasha* attributing it to *Tantrasangraha* which, in turn, attributes the result to Madhava. Similarly there is another detailed verse in *Yuktibhasha* which attributes to Madhava the infinite series expansion for arctan. The first sentence of *Yuktibhasha* states that it is going to explain all the mathematics "... useful in the motion of heavenly bodies following *Tantrasangraha*" but actually does much more as an independent treatise than work merely as a *bhasya*. (It is also remarkable for being written in Malayalam – not Sanskrit – and is in prose, not poetry.) Unlike *Tantrasangraha*, this work provides detailed argumentation for the results and is written in a style which is straightforward and unpretentious. (An English translation with detailed commentary is now available [1]).

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Here, I will concentrate on two specific results in *Yuktibhasha* and highlight the role of calculus, as we call it today, in it. Let us begin with the series expansion for $\pi/4$. To understand what is involved in obtaining the infinite series expansion for $\pi/4$ given above, it is useful to recall the procedure in modern notation. If $t = \tan\theta$

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with $0 \le \theta \le (\pi/4)$, then $(\mathrm{d}t/\mathrm{d}\theta) = 1 + t^2$. Rewriting this equation by taking the reciprocals (which is actually a rather insightful step!) and using the geometric series expansion, one gets $(\mathrm{d}\theta/\mathrm{d}t) = 1 - t^2 + t^4 + \cdots$. Integrating both sides between 0 and 1 will give you $\tan^{-1}(1) = \pi/4$ as an infinite series. Of course, the *indefinite* integral also gives the infinite series expansion of θ in terms of $\tan\theta$. It is clear that the derivation involves the concepts of differentiation and integration in some form as well as the knowledge of indefinite integrals of powers. How did the ancient mathematicians do this?

The basic idea was to consider the geometry of the circle and give meaning to all the relevant quantities in terms of suitably defined geometrical constructions. In *Figure* 1, AB is a *unit* tangent at A, and P is an arbitrary point on AB so that the length AP gives $t = \tan\theta$. Our problem then reduces to finding the length of the arc AP′ corresponding to the length AP which will give us θ as a function of t. *Yuktibhasha* begins by dividing the line AB into a large number n of equal segments marking out $\mathrm{A}_0 = \mathrm{A}, \mathrm{A}_1, \mathrm{A}_2, ..., \mathrm{A}_n = \mathrm{B}$ with the line OA_i intersecting the arc at Q_i. The key insight is to realize that as the half chord of a segment gets smaller and smaller, its length approaches the arclength. In modern

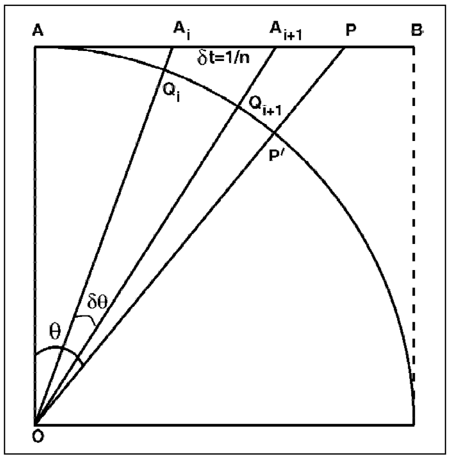

Figure 3. Construction used in *Yuktibhasha*.

language this corresponds to the realization $\sin(\theta/n) \to \theta/n$ as $n \to \infty$. There is an explicit statement in *Yuktibhasha* to this effect: "If the segments of the side ... are very very small, these half chords will be almost the same as arc segments." With this realization, as well as some very innovative geometrical reasoning, one obtains the differential relationship $\delta\theta = [\delta t/(1+t^2)]$ between $\delta t = (1/n)$ (given by $\mathrm{A}_i\mathrm{A}_{i+1}$) and $\delta\theta$ (given by $\mathrm{Q}_i\mathrm{Q}_{i+1}$) which is the relevant arc length in modern notation.

"If the segments of the side ... are very very small, these half chords will be almost the same as arc segments."

To get θ we now need to sum the series $(1 + i^2/n^2)^{-1}$ from $i = 1$ to n and finally take the limit $n \to \infty$. *Yuktibhasha*, of course, cannot do this directly and hence resorts to expanding the right-hand side into a series:

$$\delta\theta_i = \frac{1}{n} - \frac{i^2}{n^3} + \frac{i^4}{n^5} - \cdots . \tag{2}$$

(Interestingly enough, this is done by a powerful technique of recursive refining – called *samskaram*, in *Yuktibhasha*.) This reduces the problem to one of finding the sums of powers of integers. While the sum of i^k was known already to Aryabhata for $k = 1, 2, 3$, one did not know the result for higher values of k. Once again Madhava uses the fact that he only needs the result for large n and *any sub-dominant term can be discarded*. This limiting process is done with great care in *Yuktibhasha*, finally leading to the result (called *samkalitam*, or 'discrete integration') which in modern notation gives

$$\lim_{n\to\infty} \frac{1}{n^{k+1}} \sum_{i=1}^{n} i^k = \frac{1}{k+1} . \tag{3}$$

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This result (equivalent to integrating x^k and obtaining $x^{k+1}/(k+1)$) was obtained using a procedure which reduces the sum of k-th powers to the sum of $(k-1)$-th power, a technique we now call mathematical induction. From reading the relevant section of *Yuktibhasha* it is clear that its author was fully aware of the difference

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between this method of proof and the more traditional approaches usually based on geometrical reasoning; in fact, this is probably the first known instance of proof by mathematical induction in Indian mathematics. With this step completed, the result for the series can be obtained fairly easily. The clarity of the exposition regarding ideas like limiting process and infinitesimals shows how adept Madhava was at manipulating these to achieve his goal.

Equally – if not more – innovative is the approach used to obtain the series for sines and cosines. The first part of the analysis is again geometrical and closely parallels the discussion given above. By similar reasoning one can obtain the differential (again in modern notation) $\mathrm{d}(\sin\theta) = \cos\theta\,\mathrm{d}\theta$. In fact this is done with a clever geometrical construction leading to a discretized difference formula for sine and cosine of the form

$$\delta s_i \equiv s_{i+1} - s_{i-1} = 2s_1c_i; \quad \delta c_i \equiv c_{i+1} - c_{i-1} = -2s_1s_i\,, \tag{4}$$

where $s_i \equiv \sin(i\theta/2n)$ and $c_i \equiv \cos(i\theta/2n)$ for an angle θ divided into $2n$ equal parts. Unlike the case of the arctan series, one cannot now integrate this equation because the derivative of $\sin\theta$ itself is not a simple function to handle. The way *Yuktibhasha* handles this situation is absolutely ingenious.

It converts the two coupled first order differential equations for sine and cosine into a second order difference equation by introducing the quantity $\delta^2 f_i \equiv \delta f_{i+1} - \delta f_{i-1}$, where f_i is s_i or c_i and obtaining the result $\delta^2 f_i = -4s_1^2 f_i$. This is, of course, nothing but the differential equation $f'' + f = 0$ in discretized form. The next step is to solve the discretized equation by adding up the second differences to get first differences and adding up the first differences to get the function itself. This is a discretized version of the fundamental theorem of calculus and again requires careful handling of limits. The

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relevant sections of *Yuktibhasha* dwell on this important issue showing clearly that the author was aware of these subtleties.

This leads to a *samkalitam* which, in the first instance, produces what we would now call an integral equation for $\sin\theta$ in the discretised form. This equation is solved by another finite *samkalitam* by repeated substitution of the equation into itself. which requires calculating a set of sums of sums (*samkalitasamkalitam*, as you might have guessed the name should be!) defined recursively by $S_k(i) \equiv \sum_{j=1}^{i} S_{k-1}(j)$ with appropriate boundary conditions. The result for these sums of sums is given in *Yuktibhasha* but this is one instant in which the proof is not provided.

This leads to a *samkalitam* which, in the first instance, produces what we would now call an integral edsquation for sin ϕ in the discretised form.

All this is done keeping a finite n; in fact *Yuktibhasha* generally uses the technique of delaying the step of taking the limit as much as possible. After the relevant sums are done, the time is ripe to make n tend to infinity and the k-th term in the expansion of the sine series pops out to have the right value $(-1)^k\theta^{2k+1}/(2k+1)!$ – and one obtains the infinite series expansion for sine. In modern langauage what has been achieved is equivalent to converting the differential equation for $\sin\theta$ into an integral equation of the form

$$\sin\theta = \theta - \int_0^\theta \mathrm{d}\phi \int_0^\phi \mathrm{d}\chi\, \sin\chi \qquad (5)$$

and iterating it repeatedly to give

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \cdots + (-1)^k \int_0^\theta \mathrm{d}\phi_1 \cdots \int_0^{\phi_{2k-1}} \mathrm{d}\phi_{2k} \sin\phi_{2k}\,. \qquad (6)$$

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During the entire process, *Yuktibhasha* does not lose sight of the need for taking $n \to \infty$ limit in the end. The word used in this context is *sunya-prayam* which

Box 1. A Piece of Pi

Estimating the circumference of a circle of given radius has intrigued all the ancient geometers and astronomers. Most of them used the procedure of inscribing and circumscribing the circle by a polygon of a large number of sides and approximating the circumference between the perimeters of the two polygons. This, of course, requires one to find the side of the polygon as a function of the radius and consequent development of trigonometry.

While the approximate value of $\pi \simeq (22/7)$ was available since antiquity, better results were known to many ancient mathematicians. Archimedes (287–212 BC) used a polygon of 91 sides to get $3\frac{10}{71} < \pi < 3\frac{10}{70}$. Aryabhatiiya (500 AD) gives the "... circumference of a circle of diameter 20000 is 62832" getting $\pi \approx 3.1416$ while Bhaskara ($\sim$ 1114–1185) gives the circumference to be 3927 for a circle of diameter 1250 obtaining $\pi \approx 3.14155$.

What is remarkable about *Yuktibhasha* is that it does not try to give yet another approximate value for π but an *exact* infinite series expansion – which is a giant conceptual leap forward. In fact, in addition to the result described in the text, *Yuktibhasha* also gives an algebraic recursive method for finding π. In modern notation this corresponds to using the recursion relation $x_{n+1} = x_n^{-1}\left(\sqrt{1+x_n^2}-1\right)$ with $x_0 = 1$ and obtaining π as the limit

$$\pi = 4 \lim_{n\to\infty} 2^n\, x_n\,. \qquad \text{(a)}$$

This method, however, requires evaluation of square roots which is not an easy procedure while the series expansion given in the text has no square roots.

has a literal translation of 'being similar to zero'; today, we would have called it infinitesimal! There is also the usage of the word *yathestam* meaning 'as one wishes' while, for example, talking about dividing lines into arbitrarily large subdivisions.

Yuktibhasha also obtained the surface area and volume of the sphere by integration of the infinitesimal elements. For example, surface area of a sphere of radius R is computed by considering a small strip between circles of latitudes θ and $\theta + \mathrm{d}\theta$ in, say, the upper hemisphere of the sphere. The area of this strip is shown to be equal to $\mathrm{d}A = (2\pi R\sin\theta)(R\mathrm{d}\theta)$ by very careful and meticulous reasoning. In the discretised version $\delta\theta$ equals $\pi/2n$ and $\theta = i\pi/2n$ with $i = 0, 1, 2, ..., 2n$ and the total area is obtained by summing over all strips labelled by i and then taking the limit of $n \to \infty$. This is essentially the same as finding the integral of $\sin\theta$ as $\cos\theta$. Though it was known that first difference of cosine is sine, this summation is done by effectively going through the original steps once again.

Historians are still unsure whether the developments on the banks of Nila influenced corresponding later developments in the west and – as far as we know – there is no direct evidence that it did. (Kerala, of course, had direct linkage with the west, for example through trading, from very ancient days.) It is also somewhat unclear how the Nila tradition died down after the seventeenth century though several obvious historical factors can be identified as causes for its demise.

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Acknowledgement

I have benefitted significantly from several discussions with P P Divakaran on this subject.

Suggested Reading

[1] **K V Sarma, *Ganita-Yukti-Bhasa of Jyesthadeva*, Hindustan Book Agency, 2008.**

[2] **P P Divakaran, Notes on Yuktibhasha: The birth of calculus, *Indian Journal of History of Science*, 2011 [in press]; Calculus in India: The historical and mathematical context, *Current Science*, Vol.99, p.8, 2010.**

[3] **S G Rajeev, Neither Newton nor Leibnitz: The pre-history of calculus in medieval Kerala, *Lectures at Canissius College*, Buffalo, New York, 2005, available at http://www.river-valley.com/tug**

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