From Hipparchus to a Helix Ruler

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This paper gives a brief history of the chord function – a precursor of the sine function – from the mathematics/astronomy of Hipparchus and Ptolemy to the early twentieth century.

1. Introduction

Figure 1 shows an old ruler. You may be curious about the 'CHO.' scale in the middle. If you looked at it closely you may have noticed that the scale is in degrees and that the ten-degree segments become smaller as the number of degrees increases. You may have thought of the word 'chord'.

Going a step further, you may have sketched a diagram similar to that in *Figure* 2, which shows the chord CB subtended by $\angle$ CAB $= \alpha$ in a circle with radius AB $= r$. Using $ch(\alpha)$ to denote the length of the chord CB subtended by α, you may have bisected the angle CAB and noticed that

$$\sin(\tfrac{1}{2}\alpha) = \frac{\mathrm{DB}}{\mathrm{AB}} = \frac{\frac{1}{2}ch(\alpha)}{r}\,.$$

The chord function is closely related to the sine function.

Solving for $ch(\alpha)$,

$$ch(\alpha) = 2r\sin(\tfrac{1}{2}\alpha)\,. \tag{1}$$

Equation (1) shows that the chord function is closely related to the sine function. If we take the radius r equal to 1 or 10, calculations with the ch function are simplified. Early astronomers used $r = 60$, which, in the base 60 system used by the Babylonians and inherited from them by Greek astronomers, is analogous to our base 10.

Keywords
Chord function, Babylonian, helix ruler, Hipparchus, Ptolemy, Almagest, epicycle, deferent, trignometry.

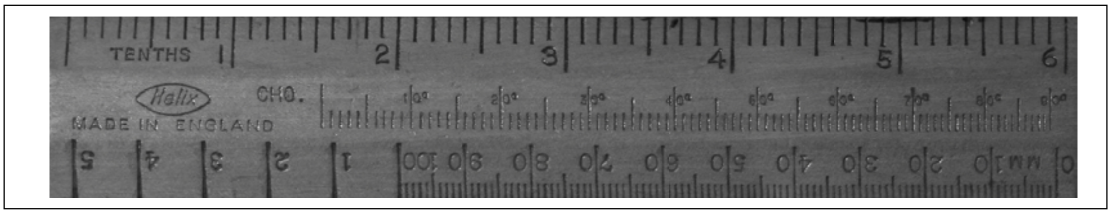

Figure 1. Helix ruler with 'CHO.' scale.

2. The Chord Function in the 20th Century CE

In searching for references to 'chords' we found the book *Practical Mathematics for Home Study...* (see [7]). In a section titled 'Angles by Chords' the author states that "An angle can usually be laid off more accurately by means of a table of chords than by the protractor, and the method is often more convenient. Although with the protractor it is not easy to measure more accurately than to half degrees, with the table of chords the angle may be laid off to tenths of a degree."

The appendix to this book includes a table of "the lengths of the chords in a circle of radius 1, for central angles from 0° to 89.9°". From the table,

$$ch(27.6^\circ) \approx 0.477\,.$$

From (1) and a modern calculator,

$$ch(27.6^\circ) = 2 \cdot 1 \cdot \sin(\tfrac{1}{2} \cdot 27.6) \approx 0.47707\,. \qquad (2)$$

The length $ch(27.6^\circ)$ of this chord in a circle of radius 1 can be approximated with the Helix ruler shown in *Figure* 1. Noting from (1) that $ch(60^\circ) = 1$ in a circle with radius 1, we found, on the actual Helix ruler, the length from 0° to 60° to be 224 points[1]. We then measured the length from 0° to 27.6° and found 108 points. The calculation of $ch(27.6^\circ)$ is then

$$\frac{108}{ch(27.6^\circ)} = \frac{224}{1} \Longrightarrow ch(27.6^\circ) \approx 0.48\,.$$

Recall from (2) that $ch(27.6^\circ) \approx 0.47707$.

[1] A printer's point is about 1/72 of an inch.

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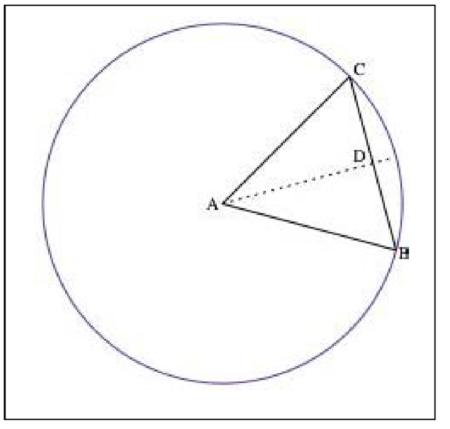

Figure 2. Chord BC in a circle.

3. The Chord Function in 150 BCE–200 CE

The chord function was an essential part of the beginnings of trigonometry around 150 BCE, some 150 years after Euclid compiled his *Elements*. The Greek astronomer Hipparchus (190–120 BCE) is generally credited with calculating the first chord table[2]. Hipparchus needed such a table for calculations related to his theory of the motion of the moon.

[2] B L van der Waerden argues in [2] that this credit should go instead to Apollonius of Perga (262–190 BCE).

Most of Hipparchus' writings have not survived, except as used and extended in Ptolemy's *Almagest* [3]. Claudius Ptolemaeus (100–175 CE), who lived and worked in Alexandria in 150 CE, was also an astronomer. His *Almagest* gives a comprehensive exposition of the mathematical astronomy of the Greeks and is comparable to Euclid's *Elements* in its subsequent influence and use.

[3] Ptolemy's table lists the chords for angles from ½º to 157½º.

We give a short extract from Ptolemy's chord table, note a few trigonometric identities that Ptolemy used, and work though one of his astronomical calculations.

The *Almagest* gives a comprehensive exposition of the mathematical astronomy of the Greeks, and is comparable to Euclid's Elements.

Table 1 contains four columns. The first two columns were taken directly from the *Almagest*. Thinking of 'Arcs' as 'angles', the first column lists the angles[3] from $\frac{1}{2}^\circ$ to $10\frac{1}{2}^\circ$. The second column lists, in a circle of radius 60, the lengths of the chords corresponding to these angles.

Arcs	Chords (Ptolemy)	Chords (Base 60)	Chords (Base 10)
$\frac{1}{2}$	0;31,25	0;31,24,56,59	0.5235971
1	1;2,50	1;2,49,51,48	1.0471843
$1\frac{1}{2}$	1;34,15	1;34,14,42,19	1.5707515
2	2;5,40	2;5,39,26,22	2.0942888
$2\frac{1}{2}$	2;37,4	2;37,4,1,49	2.6177862
3	3;8,28	3;8,28,26,30	3.1412338
$3\frac{1}{2}$	3;39,52	3;39,52,38,16	3.6646216
4	4;11,16	4;11,16,34,57	4.1879396
$4\frac{1}{2}$	4;42,40	4;42,40,14,25	4.7111779
5	5;14,4	5;14,3,34,31	5.2343265
$5\frac{1}{2}$	5;45,27	5;45,26,33,5	5.7573754
6	6;16,49	6;16,49,7,59	6.2803147
$6\frac{1}{2}$	6;48,11	6;48,11,17,3	6.8031345
7	7;19,33	7;19,32,58,9	7.3258247
$7\frac{1}{2}$	7;50,54	7;50,54,9,7	7.8483755
8	8;22,15	8;22,14,47,48	8.3707768
$8\frac{1}{2}$	8;53,35	8;53,34,52,4	8.8930188
9	9;24,54	9;24,54,19,46	9.4150915
$9\frac{1}{2}$	9;56,13	9;56,13,8,44	9.9369849
10	10;27,32	10;27,31,16,51	10.4586891
$10\frac{1}{2}$	10;58,49	10;58,48,41,57	10.9801942

Table 1. A short extract from Ptolemy's 'Chord Table'.

Ptolemy gave the lengths of the chords in base 60. The first entry, for example, means

$$0;31,25 = 31/60 + 25/60^2 \approx 0.523611\,.$$

The semicolon separates the integer part of the number from the fractional part. The use of base 60 by both Hipparchus and Ptolemy in their astronomical tables and their calculations is evidence of some influence from Babylonian sources.

The use of base 60 by Hipparchus and Ptolemy is evidence of some transmission from Babylonian sources.

To explain the motion of the planets, Ptolemy used a geometric model based on epicycles.

The third and fourth columns can be calculated using (1), with $r = 60$. The third column gives accurate values of these chords in base 60. Note that, to the number of base 60 places displayed, Ptolemy's values are mostly correct. The fourth column gives the lengths of the chords in base 10.

Ptolemy calculated his chord table knowing the chords of 36°, 60°, 72°, 90°, and 120°, which correspond to the sides of the regular polygons with 10, 6, 5, 4, and 3 sides. He also knew the equivalents, in terms of the chord function, for the trigonometric identities $\sin(180° - \alpha)$, $\sin(\alpha - \beta)$, $\sin(\alpha + \beta)$, and $\sin(\frac{1}{2}\alpha)$.

An interesting discussion of Ptolemy's chord table calculations can be found in Chapter 4 of [4].

4. An Astronomical/Trigonometrical Calculation from the *Almagest*

To explain the motion of the planets as seen from the earth, Ptolemy used a geometric model based on *epicycles*. See *Figure* 3.

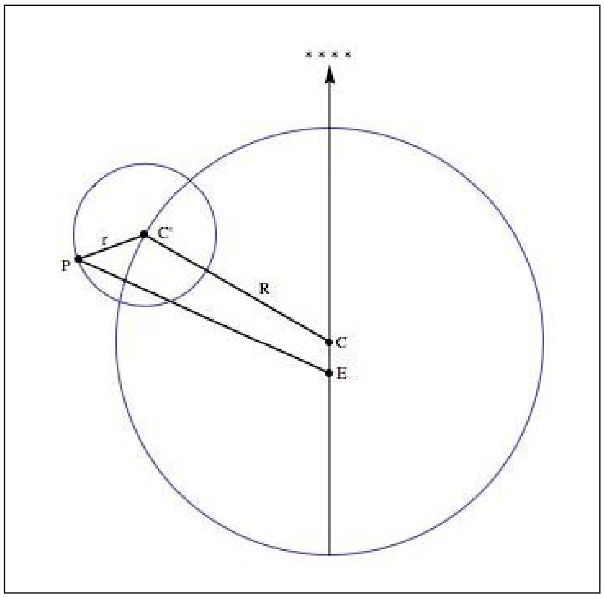

Figure 3. The motion of planets according to the theory of epicycles.

The plane of the two circles shown in the figure is the plane of the ecliptic[4]. The planet P moves counterclockwise on the the small circle about its center C′. This circle is called an *epicycle*. The center C′ of the epicycle moves clockwise about the center C of the larger circle, the *deferent*. The line through the observer at E and the center C of the deferent is fixed relative to the stars.

[4] The ecliptic is the apparent path of the sun as seen from the earth, relative to the fixed stars.

In practice, Ptolemy refined this model in several ways. The resulting model gave accurate predictions of planetary motion.

We conclude with an example of Ptolemy's use of his chord table in solving a problem from the *Almagest*. This is adapted from pages 7–9 and 163–164 from Toomer's [3] translation and commentary on the *Almagest*, starting with "Similarly,…" on page 163. We alternate 'Discussion' with Ptolemy's arguments. We occasionally rephrase Toomer's translation in an attempt to clarify the discussion.

Discussion: In *Figure* 4, $\angle$ TAH $= 30°$, AD $= 60$, and AH $= 2;30$ are given. The line KH is perpendicular to AD. Ptolemy wishes to calculate $\angle$ HDK.

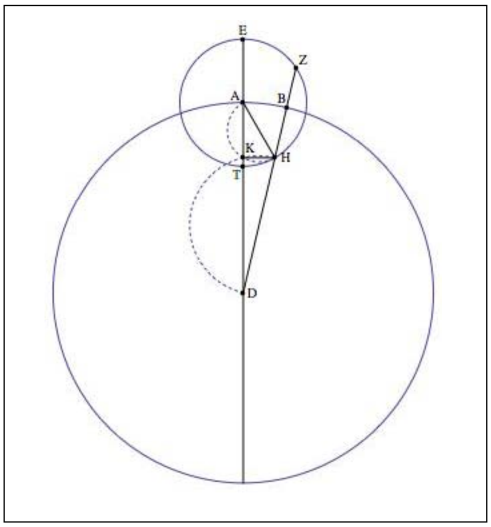

Figure 4. A figure consistent with Ptolemy's calculations.

Ptolemy imagines semicircles about right triangles HKA and DHK. These constructions – based on Euclid's Proposition 20, *Book* III – are shown in *Figure* 4 as two 'dotted' semicircles[5]. The lines KH, KA, and DK are now chords in at least one of these semicircles and can therefore be calculated using a chord table. Recall, however, that the chord table is based on a circle with diameter 120. In the next step, Ptolemy therefore assumes, temporarily, that the length of AH is 120 units.

[5] Neither the two semicircles nor the point L are present in the figures accompanying Toomer's translation or commentary.

Ptolemy: Taking AH = 120,

$$\begin{aligned} \mathrm{HK} &= 60 \\ \mathrm{AK} &= 103;55\,. \end{aligned}$$

Discussion: Because triangle KLH is equiangular, HK = 60. Also, ∠ ALK = 120°. From (1),

$$\mathrm{AK} = ch(120^\circ) = 120\sin(\tfrac{1}{2}\cdot 120) = 60\sqrt{3} \approx 103;55\,.$$

In his next step, Ptolemy goes back to the original lengths of AH and AD.

Ptolemy: Therefore if AH = 2; 30 and radius AD = 60,

$$\mathrm{HK} = 1;15,\ \ \mathrm{AK} = 2;10,\ \text{and,}$$

$$\text{by subtraction}, \mathrm{KD} = 57;50\,.$$

Discussion: Using similarity of triangles,

$$\begin{aligned} \frac{\mathrm{HK}}{2;30} &= \frac{60}{120} \implies \mathrm{HK} = 1;15 \\ \frac{\mathrm{AK}}{2;30} &= \frac{103;55}{120} \implies \mathrm{AK} = 2;10 \end{aligned}$$

The arithmetic in the calculation for AK is, in base 10,

$$\begin{aligned} \mathrm{AK} &= \frac{(2+30/60)(103+55/60)}{120} \\ &= \frac{1247}{576} \approx 2.164930556\,. \end{aligned}$$

This number, converted to base 60, is approximately 2;9,53, which rounds to 2;10 and therefore agrees with Ptolemy's value for AK.

Finally,

$$\mathrm{KD} = \mathrm{AD} - \mathrm{AK} = 60 - 2;10 = 57;50\,.$$

Ptolemy: Because $\mathrm{HK}^2 + \mathrm{KD}^2 = \mathrm{DH}^2$,

$$\begin{aligned} \mathrm{DH} &= \sqrt{(1+15/60)^2 + (57+50/60)^2} \\ &= \frac{\sqrt{481861}}{12} \approx 57;51\,. \end{aligned}$$

Therefore if the hypotenuse $\mathrm{DH} = 120$,

$$\mathrm{HK} = 2;34 \text{ and } \angle\mathrm{HDK} = 2;27^\circ\,.$$

Discussion: Accepting for the moment the value $\mathrm{HK} = 2;34$, Ptolemy interpolated in his table of chords to find the angle $2^\circ 27'$.

There appears to be a minor error in Ptolemy's calculation of HK. If $\mathrm{DH} = 57;51$ and $\mathrm{HK} = 1;15$, then, using $\mathrm{DH}' = 120$ and HK' for the 'new' values of DH and KH we see by similarity that

$$\begin{aligned} \mathrm{HK}' = \frac{\mathrm{DH}'\cdot\mathrm{HK}}{\mathrm{DH}} = \frac{120\cdot 1;15}{57;51} = \frac{3000}{1157} \\ = 2;35,34,\ldots\,. \end{aligned}$$

This value is slightly different from Ptolemy's value of 2;34 for HK.

Summary

This brief discussion of the chord tables used by Hipparchus and Ptolemy gives only a glimpse of their impressive accomplishments in both mathematical astronomy and mathematics. Readers who wish to learn more may examine one or more of the items listed in the Suggested Reading. Each of these has additional references.

Suggested Reading

[1] Clarence Irwin Palmer, *Practical Mathematics for Home Study, Being the Essentials of Arithmetic, Geometry, Algebra and Trigonometry*, McGraw-Hill, New York, 1919.

[2] B L van der Waerden, Reconstruction of a Greek table of chords, *Archive for History of Exact Sciences*, Vol.38, pp.23–38, 1988.

[3] Claudius Ptolemaeus (Translated and Annotated by G J Toomer), *Ptolemy's Almagest*, Springer-Verlag, New York, 1984.

[4] Asger Aaboe, *Episodes from the Early History of Astronomy*, Springer, New York, 2001.

[5] G J Toomer, The Chord Table of Hipparchus and the Early History of Greek Trigonometry, *Centaurus*, Vol.18, pp.6–28, 1974.

[6] Asger Aaboe, *Episodes from the Early History of Mathematics*, Random House, New York, 1964.

[7] O Neugebauer, *The Exact Sciences in Antiquity*, Second Edition, Dover Publications, New York, 1969.

[8] Olaf Pedersen, A Survey of the Almagest, *Acta historica scientiarum naturalium et medicinalium*, Vol.30, 1974.

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