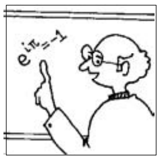
Classroom



In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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A Pedagogical Study of Cooling of a Granualar Gas

A dust devil on a hot summer day is a fascinating example of a granular gas. There is an initial stage, termed the homogeneous cooling stage (HCS), during which the grains of such a granular gas are uniformly distributed and have random velocities. A characteristic granular temperature can be associated with the kinetic energy of this random motion of the grains. The grains undergo inelastic collisions. This leads to loss of kinetic energy and the gas 'cools'. Haff demonstrated that the granular temperature during the HCS stage reduces with time in a power law fashion. We present a pedagogical treatment of this famous law of cooling.

1. Introduction

Granular materials have been a subject of intense study in recent times. These materials exhibit properties some of which are akin to solids while others are akin to liquids. For example, sand, a granular material, flows like a liquid. On the other hand it exhibits solid-like properties such as incompressibility and jamming while flowing through a constriction. Not so surprisingly granular materials also exhibit gas-like behaviour. Examples

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are: dust devils during hot dry summers (*Figure* 1) and barchan dunes in deserts on Earth (and even Mars and Venus). A granular gas can be simulated by placing identical small smooth elastic balls on a horizontal bed. The bed is then given a high vibrational acceleration. The grains then get randomly distributed and have random velocities. A demonstration of this can be seen in reference [1]. The particle collision times become small when compared to the mean free times between collisions. The fluidized granular material then approximates the classical picture of a gas. One can define a granular temperature and even attempt to obtain the equation of state for such a gas [2, 3].

Figure 1. A typical dust devil during summer.

An important feature of a granular material in gaseous state is its cooling. This occurs since the collisions between the grains are inelastic and there is loss of kinetic energy (or cooling). Initially the grains are distributed uniformly and have random velocities. The gas remains homogeneous while the mean velocities of its particles decrease continuously. This stage is termed the homogeneous cooling stage (HCS). The grains are smooth and hence friction is absent. When two grains collide the relative tangential motion between the grains are unaffected. The normal component however is reduced since the collision is inelastic. This leads to local parallelization of particle velocities and some clustering. In the clustered regions the grains undergo increasing collisions which further reduce the velocities and thereby stabilize the inhomogeneities. This is called the inhomogeneous cooling state (ICS) [4, 5]. One can show that the HCS is unstable to fluctuations in the density and velocity fields. The ICS is characterized by the emergence and growth of particle-rich clusters, with particles in a cluster moving in approximately parallel directions [6].

The HCS was first studied by P K Haff and corresponds to an approximately homogeneous state of the granular gas. Since the collisions are inelastic the random velocities of the grains reduce with time. Note that the granular temperature is a measure of random motion. It was shown by Haff that this temperature will reduce with time in a power law fashion. In the present article we present a pedagogical treatment of this famous law of cooling proposed by Haff [7].

2. The Model

We consider an isolated system of homogeneous and isotropic granular gas. Hereafter we will refer to granular gas as gas only. There are no external forces acting on this system. For simplicity we assume that this system consists of N identical particles or grains which are hard spheres with mass m and diameter σ. We assume that the spheres are smooth so that friction can be ignored between the particles. Let the total volume occupied by the gas be V. Therefore the number density n is N/V. We assume N to be a large number which allows us to use the methods of statistical mechanics. Analogous to the kinetic theory of gases, the grains have randomly distributed velocities. Their average velocity is zero, i.e., $< \vec{v} >= 0$. A relevant quantity to characterize this irregular motion is the root mean square velocity v_{rms}. Just as we define the temperature in the kinetic theory of gases, it is natural to introduce the *granular temperature* of the gas as the average kinetic energy of the grains:

$$\frac{3}{2}k_{\mathrm{g}}T = \left\langle \frac{1}{2}m\vec{v}^2 \right\rangle = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{2}m\vec{v}_i^2, \qquad (1)$$

where k_{g} is the Boltzmann constant for granular gas material and $\vec{v}_i$ is the velocity of the i-th grain. An important note regarding this temperature is that it should not be confused with the physical temperature which can be measured with the thermometer. Over a period of time, the inelastic particle collisions result in a decay in the granular temperature of the gas. We shall substantiate it with a detailed discussion in the next section.

Consider an instantaneous collision between particles i and j with velocities $\vec{v_i}$ and $\vec{v_j}$ respectively. Their velocities after the collision are $\vec{v_i}'$ and $\vec{v_j}'$ respectively. For better understanding we shall attempt the problem in $\{\hat{n},\ \hat{z}\}$ coordinates, where $\{\hat{n}\}$ is the unit vector normal to the sphere i in the plane containing the centres of particles i and j and passing through the point of contact, and the tangential unit vector $\{\hat{z}\}$ is perpendicular to $\{\hat{n}\}$ and in the same plane as shown in *Figure* 2. The velocity v_i of a typical grain labelled 'i' may thus be decomposed as

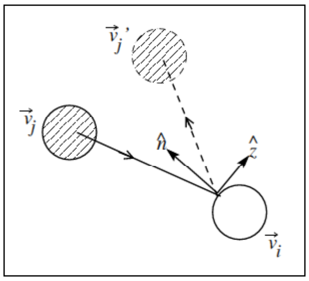

Figure 2. The inelastic collision of grain j (shaded) with grain i. See text for detailed description.

$$\vec{v_i} = v_{in}\hat{n} + v_{iz}\hat{z}\ . \tag{2}$$

Since the collision is inelastic, the kinetic energy of the system of spheres decreases. Since there is no friction between the spheres, hence after the collision the tangential component of velocity remains unchanged. Therefore the normal component of the relative velocity diminishes by a factor $e < 1$, where e is the coefficient of restitution. In other words

$$v'_{in} - v'_{jn} = -e(v_{in} - v_{jn})\ . \tag{3}$$

The elastic limit corresponds to $e = 1$. In what follows we may also assume that the motion of grains are uncorrelated so that

$$< v_i v_j v_k ... > = < v_i >< v_j >< v_k > ...\ . \tag{4}$$

3. Dynamics of a Granular Gas

We apply the law of conservation of momentum to the collision depicted in *Figure* 2.

$$m\vec{v_i} + m\vec{v_j} = m\vec{v_i}' + m\vec{v_j}'\ . \tag{5}$$

Recall that spheres are smooth and there is no force along the $\hat{z}$-axis. Therefore

$$v_{iz} = v'_{iz} \quad \text{and} \quad v_{jz} = v'_{jz} \; . \tag{6}$$

Using the above equations we can show that

$$v'_{in} = v_{in}\frac{1-e}{2} + v_{jn}\frac{1+e}{2} \; , \tag{7}$$

$$v'_{jn} = v_{in}\frac{1+e}{2} + v_{jn}\frac{1-e}{2} \; . \tag{8}$$

The change in kinetic energy can be given by $\Delta K = K_i - K_f$, where K_i and K_f are the total kinetic energy before and after the collision respectively.

$$\Delta K = \frac{1}{2}mv_i^2 + \frac{1}{2}mv_j^2 - \frac{1}{2}mv_i'^2 - \frac{1}{2}mv_j'^2 \; . \tag{9}$$

After solving and substituting the values of $\vec{v_i}'$ and $\vec{v_j}'$ from (7) and (8), we obtain

$$\Delta K = \frac{m}{4}(1-e^2)(v_{in} - v_{jn})^2 \; . \tag{10}$$

The decay of temperature ΔT during a certain time span corresponds to the loss of kinetic energy due to dissipative collisions. The total loss in kinetic energy in time Δt can be calculated as the product of average loss of kinetic energy in one collision and the average number of collisions which occur in time Δt. Hence

$$-\frac{\mathrm{d}K}{\mathrm{d}t} = < \Delta K > \nu \; . \tag{11}$$

The negative sign corresponds to the decrease in kinetic energy of the system. The average number of collisions per unit time or collision frequency ν may be estimated as follows: Consider a grain moving with average velocity $< v_r >$ relative to other grains while all other grains

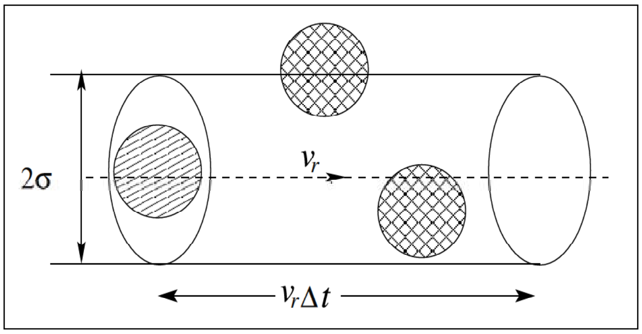

Figure 3. The collision cylinder of the grain. Only grains whose centres are located inside the collision cylinder of radius σ and length < v_r> Δt collide with the grain on the left. Here Δt is the time interval. See text for more details.

are assumed to be fixed scatterers. During the time interval Δt, this grain collides with all grains whose centres lie inside the collision cylinder of length $< v_r > \Delta t$. We may take $v_r \propto v_{\rm rms}$. The cross-section of the cylinder equals $\pi\sigma^2$, where $\sigma = 2R$ is the grain diameter as shown in *Figure* 3. Its volume is then proportional to $\sigma^2 v_{\rm rms}\Delta t$. The average number of scatterers in the collision cylinder is equal to the product of the number density n and the volume of this collision cylinder. This is also the number of collisions during Δt. Hence the collision frequency can be written as

$$\nu = \alpha n \sigma^2 v_{\rm rms}, \quad (12)$$

where α is a dimensionless constant.

Just as in the case of kinetic theory of gases we relate $v_{\rm rms}$ to granular temperature,

$$v_{\rm rms} = \beta\sqrt{T}\ , \quad (13)$$

where β is a constant. The average value of loss of kinetic energy per collision is

$$\begin{aligned} < \Delta K > &= \left\langle \frac{m}{4}(1-e^2)(v_{in}-v_{jn})^2 \right\rangle \\ &= \frac{m(1-e^2)}{4} < v_{in}^2 + v_{jn}^2 - 2v_{in}v_{jn} > . \quad (14) \end{aligned}$$

We have already pointed out in (4) that grain velocities are uncorrelated. Hence the average value of the third term on the RHS is zero. The average value of the first and second terms can be derived as follows:

$$< v_{in}^2 + v_{iz}^2 >= v_{\rm rms}^2 \ . \qquad (15)$$

The assumption of randomness allows us to equate the normal and tangential components of the velocity ($< v_{in}^2 >=< v_{iz}^2 >$). Hence we obtain

$$v_{in} = \frac{v_{\rm rms}}{\sqrt{2}} \ . \qquad (16)$$

The average loss in kinetic energy per collision is then given by

$$< \Delta K >= \frac{m(1-e^2)v_{\rm rms}^2}{4} \ . \qquad (17)$$

Going back to (1) and differentiating it, we obtain

$$-\frac{{\rm d}K}{{\rm d}t} = \frac{3}{2} N k_{\rm g} \frac{{\rm d}T}{{\rm d}t} \ . \qquad (18)$$

Using (11), (12) and (17) we obtain

$$-\frac{3}{2} N k_{\rm g} \frac{{\rm d}T}{{\rm d}t} = \frac{\alpha n \sigma^2 m (1-e^2) v_{\rm rms}^3}{4} \ . \qquad (19)$$

Thus

$$-\frac{{\rm d}T}{{\rm d}t} = \gamma v_{\rm rms}^3 \ , \qquad (20)$$

where $\gamma = \alpha n \sigma^2 m(1-e^2)/6Nk_{\rm g}$. Substituting the value of $v_{\rm rms}$ from (13) we obtain

$$-\frac{{\rm d}T}{{\rm d}t} = \gamma' T^{3/2} \ , \qquad (21)$$

where $\gamma' = \gamma\beta^3$. Integrating (21) yields

$$T(t) = \frac{4T_o}{(2+\gamma' t\sqrt{T_o})^2} \ , \qquad (22)$$

where T_o is the initial granular temperature. This expression can be cast in a physically transparent form:

$$T(t) = \frac{T_o}{(1+t/\tau_o)^2} \ , \qquad (23)$$

where $\tau_o = 2/\gamma'\sqrt{T_o}$. This is the Haff's law of cooling for HCS.

4. Conclusions

The above derivation has been carried out for inelastic collisions. We can verify our results by taking the elastic limit, i.e., $e = 1$. In this case equation (14) implies that there will be no loss in kinetic energy. Thus $\mathrm{d}T/\mathrm{d}t$ will be zero (see (19)) and the temperature will remain a constant. Our derivation yields Haff's law but note that it is not possible to obtain a numerical estimate of time constant τ_0 in (23) because of the uncertainties associated with the constants α, k_{g}, etc.

At this point it is interesting to see the results for granular materials consisting of viscoelastic particles. These particles have a coefficient of restitution which depends on the relative impact velocity. For example e is found to vary in the following manner [8]:

$$e \simeq 1 - c_1 v_r^{1/5} \ ,$$

where c_1 is a small constant and v_r is the relative velocity. Once again we take $v_r \propto v_{\mathrm{rms}}$. Ignoring the second order term in c_1 and using (13), we obtain

$$1 - e^2 \propto v_{\mathrm{rms}}^{1/5} \propto T^{1/10}.$$

For this case, (21) can now be rewritten as

$$-\frac{\mathrm{d}T}{\mathrm{d}t} = \gamma_1 T^{8/5} \ ,$$

where $\gamma_1 \propto \alpha n \sigma^2 m/6Nk_{\mathrm{g}}$. Haff's cooling law for such grains will then be

$$T(t) = \frac{T_0}{(1 + t/\tau_0')^{5/3}} \tag{24}$$

with $\tau_0' = 5/3\gamma_1\sqrt{T_0}$. Thus the cooling of viscoelastic grains is slower than grains with constant e (*Figure* 4).

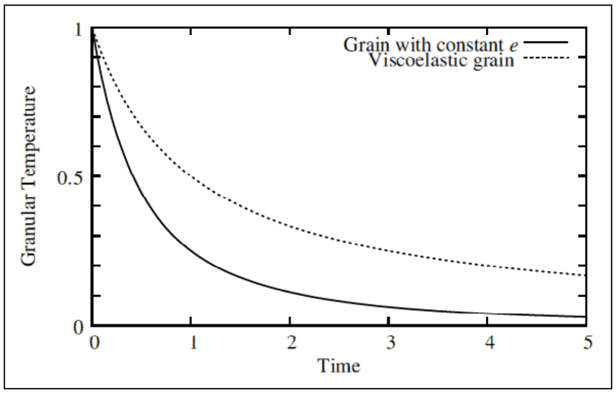

Figure 4. A relative comparison of variation of granular temperature with time for grains with constant e and viscoelastic grains (equation (23) vs equation (24)). Here we have taken the values of τ_0 and τ'_0 to be unity and the x-axis represents normalized time.

Many attempts have been made to verify the Haff's law [4, 5]. A cylindrical system containing identical grains is placed on an electrodynamic shaker. At high acceleration the grains behave like the molecules of a classical gas. The coefficient of restitution is high ($e = 0.92$). A radioactive nucleus is embedded in a grain and its trajectory is monitored with a Geiger tracer. A detailed analysis is done to calculate the granular temperature. An alternative procedure is to investigate cooling by a molecular dynamics simulation. It is found that the temperature decays as t^{-n}, where $n = 2.2$, slightly higher then Haff's prediction. Note that we have neglected friction and packing fraction.

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Suggested Reading

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