Dawn of Science

17. Geometry Without Figures

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Resonance, Vol.15: p.498, p.590, p.684, p.774, p.870, p.1009, p.1062. Vol.16: p.6, p.110, p.274, p.304, p.446, p.582, p.663, p.770, p.854.

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Descartes, analytic geometry, Fermat, Wiles.

Descartes discovered the link between algebra and geometry, which forms the cornerstone of applied geometry today.

There is a story about Ptolemy Soter, the first king of Egypt and founder of the Alexandrian Museum, who studied geometry under Euclid. He found the subject rather difficult and apparently asked his teacher whether there is an easier way of learning geometry. To which he received the famous reply: "There is no royal road to geometry."

In a sense, Euclid was wrong. There is a way of doing geometry using algebra which is considerably simpler and conceptually more straightforward. The discovery of this connection between algebra and geometry was definitely a milestone in science.

One person who contributed most to this subject was René Descartes (1596–1650), the French philosopher and mathematician. (Descartes used to sign in the Latinized version of his name 'Cartesins', because of which both his systems of geometry and philosophy go under the name 'Cartesian'.)

Descartes was born in 1596 in France. He was a brilliant but very unhealthy student and obtained from his teachers the concession to remain in bed, as long as he wished, everyday – a habit which he continued to his adult years. His early Jesuit education made him extremely devout and faithful. In 1633, when he heard about Galileo's fate, he abandoned the idea of writing a book in support of the Copernican theory. Instead, he came out with an astronomical model in which the Earth was in the centre of a 'cosmic vertex' with the vertex travelling around the Sun. Though many people accepted this compromise, it was worthless as an astronomical model.

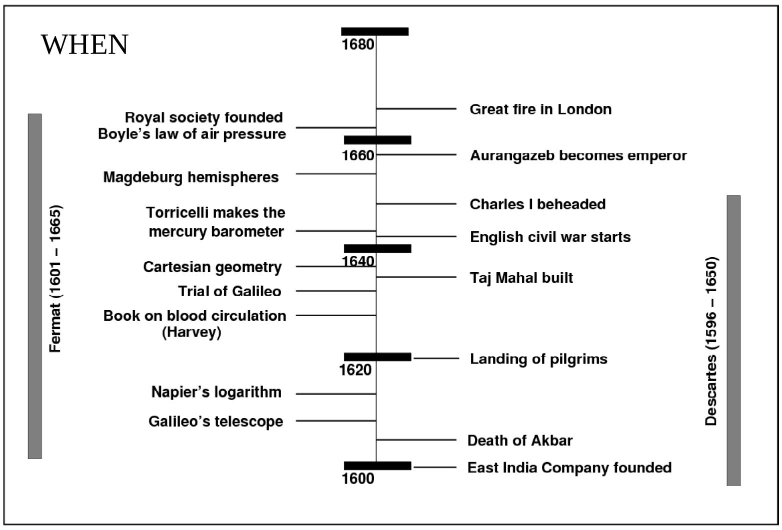

Figure 1.

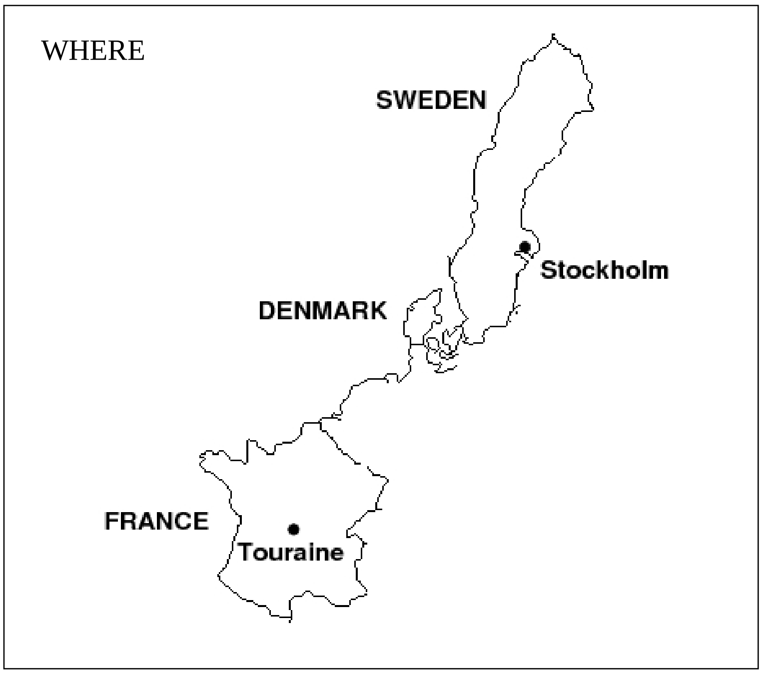

Figure 2.

After his education, Descartes joined the French army. Fortunately, he was never exposed to actual warfare and hence had a lot of spare time to work out his ideas. It was during this time that Descartes seems to have stumbled upon an important discovery, from which originated the branch of 'analytic geometry'. The

Figure 3. Descartes.
Courtesy:
http://en.wikipedia.org/wiki/Rene_Descartes

basic idea was extremely simple. It was well known (right from the days when maps were used) that the position of a point on a two-dimensional surface could always be represented by two numbers. For example, the location of any city on the Earth could be specified by the latitude and the longitude. Similarly, the position of any point on this paper could be denoted by giving its distance from the bottom edge of the paper and from the left end. Descartes realized that this fact allowed any curve in geometry to be represented by an algebraic equation. For example, consider the equation $y = x^2$. If we now give for x a series of values like 1, 1.5, 2, 2.5, etc., we will obtain for y the values 1, 2.25, 4, 6.25, etc., respectively. Each pair of these numbers – that is, (1, 1), (1.5, 2.25), (2, 4), (2.5, 6.25) – represents a point in the plane. By connecting these points, we obtain a smooth curve, which is unique. We have thus coded the information about the geometrical curve into the algebraic equation $y = x^2$. We can now methodically translate all the basic geometrical relations concerning the plane figures into equivalent algebraic statements. Once this is done, any property of a geometrical figure can be

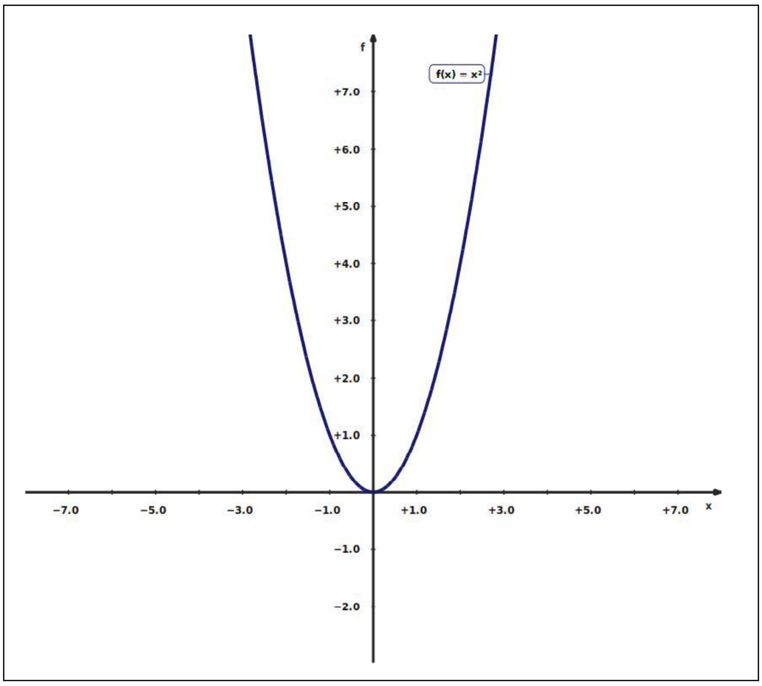

Figure 4. A correspondence can be established between any function, say, $f = x^2$, and a curve in the plane. The function provides a set of pairs of numbers; in the case of $f(x) = x^2$, $f = 0$ at $x = 0$; $f = 1$ at $x = 1$; $f = 4$ at $x = 2$; $f = 9$ at $x = 3$, etc. These sets of points can be connected by a smooth curve. Similarly, any geometric curve can be expressed as an algebraic function.

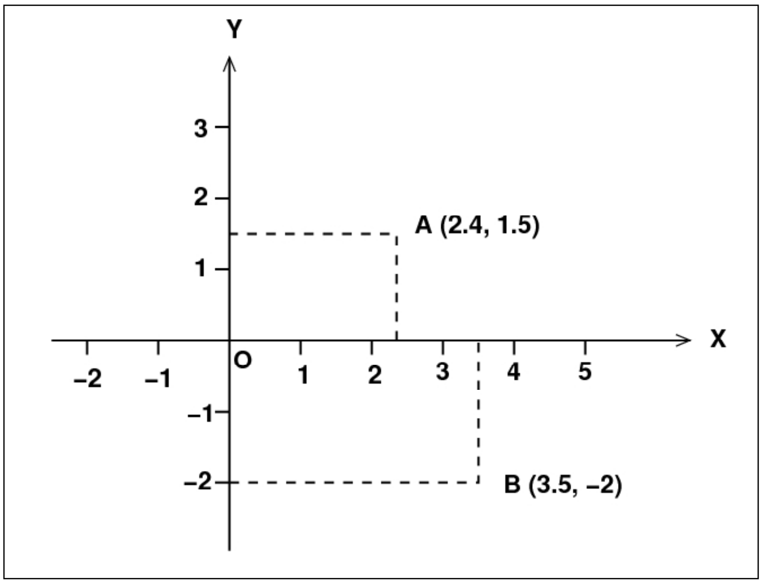

Figure 5. Any point on a plane can be specified by giving two numbers. We first draw reference lines (OY, OX) and mark them at equal intervals. Distance to the 'left' or 'below' are treated as negative. The point A can be specified by its distance from OY (2.4 units) and its distance from OX (1.5 units), in that order. Similarly, to denote a point below the OX axis (say B), we specify the corresponding distance with a negative number; as –2 in this case.

worked out using purely algebraic operations without ever drawing the figure.

The connection described above is extremely powerful and forms the cornerstone of applied geometry today. Its power stems from two facts. First, this connection between algebra and geometry provides a systematic procedure for deriving and proving geometrical relations; for example, suppose we have to prove that the area of a triangle with sides a, b, c is the square root of the expression $s(s-a)(s-b)(s-c)$, where s is half the perimeter of the triangle. In conventional geometry, there is no obvious way of proving such a result. One has to draw the figure of the triangle, introduce clever constructions and use appropriate theorems to arrive at this result. Analytic geometry provides a *brute-force* method for tackling the same problem. We can represent the three vertices of the triangle by three pairs of numbers : (x_1, y_1), (x_2, y_2), (x_3, y_3). By using the rules of analytic geometry – and without requiring much creativity or imagination – one can write down the expression for the area of a triangle and, comparing it with the given expression, one can easily prove the result.

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extends enormously the range of geometrical curves and shapes that can be studied. Conventional Greek geometry was restricted to straight lines, circles, conic sections and some simple extensions of these. But now we see that every equation of the form $y = f(x)$ represents some curve in the plane whose properties can be studied. In this sense, algebra has enriched the scope of geometry.

Descartes was led to this discovery through his mechanistic view of the world. He tried to think of all phenomena in terms of the motion of mechanical gadgets; to him, a geometrical curve was essentially a path traced by an object moving in a particular way. Descartes published his results as the last of three appendices accompanying his treatise, *Discours de la methode....* (*Discourse on the method*) which dealt with the proper way of conducting logical study in various branches of science. Most of the material in the treatise is trivial and useless; however, the third appendix contains the real gem. (It is rather amusing to note that there has been at least another occasion in mathematics when the appendix was more important than the text. In 1831, several key ideas in non-Euclidean geometry appeared as an appendix in a book by Bolyai!)

Suggested Reading

[1] E T Bell, *Men of Mathematics*, (Touchstone) Simon and Schuster, 2008.

[2] Howard Eves, *Great Moments in Mathematics / Before 1650*, The Mathematical Association of America, 1983.

After his service in the army, Descartes returned to protestant Holland where he stayed almost till the end of his life. During this period Queen Christina ruled Sweden and she was one of the most eccentric queens who ever lived. It had become fashionable in Europe at this time for the royalty to invite intellectuals to their courts and pretend a keen interest in matters of the mind. Following this trait, Christina invited Descartes to the Swedish court, which he – unfortunately – accepted. Christina made Descartes call on her three times each week, at five in the morning, during one of the worst Swedish winters. It is not known whether the queen grew in intellect as a result of this exercise; poor Descartes, however, caught pneumonia and died. In protestant Sweden, he was buried in a cemetery reserved for unbaptized children, because he was a Catholic. In 1667, his remains were carried to Paris and buried again – but not with much pomp and splendour since it was thought that his views were 'too hot'. Finally, he was

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Box 1. Fermat's Last Theorem

Fermat (1601–1665) was a contemporary of Descartes and has been often called the 'prince among amateur mathematicians'. A counsellor at the French Parliament, he devoted much of his spare time to mathematics. He made contributions in developing the theories of probability and analytic geometry and had some notion of calculus. But what probably made him 'immortal' was his contributions to the theory of numbers – in particular, a note he scribbled on the margin of a book (*Arithmetica* of Diophantus).

This marginal note said that an equation of the form $x^n + y^n = z^n$ cannot be satisfied with x, y, z and n being integers with $n > 2$. (When $n = 2$, there are several solutions like $3^2 + 4^2 = 5^2$ and $5^2 + 12^2 = 13^2$, etc.). Fermat added that "I have discovered a truly marvellous proof of this, which, however, the margin is not large enough to contain." Many a later mathematician has wished that *Arithmetica* had a wider margin!

Fermat had also stated several other 'theorems' in his correspondence and in all but one case, he was proved right (in one case he was proved wrong). Only the 'last theorem' quoted above remained for a long time without being proved right or wrong. A tremendous amount of research went into investigating this theorem and new vistas of mathematics have originated from these studies. This theorem also led to dozens of false proofs year after year. Since the statement of the theorem is so simple, it is accessible to anyone who has a basic education in algebra. Many amateurs try their hand at it without even appreciating the true problem. To be fair to them, it must be said that even some professional mathematicians have been guilty of publishing wrong proofs!

In June 1993, Andrew Wiles presented what he considered to be the proof of Fermat's Last Theorem using fairly sophisticated mathematical techniques. Interestingly enough, a critical portion of the proof contained an error which was caught by several mathematicians refereeing Wiles' manuscript. Wiles, along with his former student Richard Taylor, had to spend nearly another year to take care of this difficulty and the final result was submitted to journals in the form of two papers (one by Wiles and the other co-authored with Taylor) in October 1994, about 358 years after Fermat first conjectured it!

disinterred during the French Revolution and was buried once again with other French thinkers in the Pantheon.

Almost at the same time, there was another French man who developed parts of analytic geometry independently. This was Pierre de Fermat (1601–1665), a parliament counsellor who devoted only his spare time to mathematics. He worked in mathematics 'for fun' and developed the frustrating habit of stating theorems in the margins of books and in private correspondence. He developed analytic geometry in two and three dimensions around 1630 but never bothered to publish it. We only know of this fact from a letter he wrote to his friend in 1637.

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