Matrix Models –
An Approach to Understand Complex Systems

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Matrices with random matrix elements appear to have applications in physics, mathematics, biology, telecommunications, etc. In fact, experimental data of many complex systems, such as the spacing distribution of energy level spectra of heavy nuclei, and the distribution of the non-real zeros of the Riemann zeta function can be reproduced using the theory of such matrices. In this article, we present the integral calculus of matrices with random matrix elements which will provide the necessary foundation to study their applications. In particular, we will present in detail an elegant and systematic tool called Feynman’s method to perform integration of matrices.

1. Introduction

It is well known that any realistic system involves complicated many-body interactions. In most complex situations, neither the energy operator \( \hat{H} \) (Hamiltonian) nor the energy eigenvalues are known. For such many-body interacting systems, we are only interested in the average properties. Such a description is broadly classified as ‘statistical mechanics’ where we look at an ensemble (large collection of identical systems) and study the macroscopic properties (also called statistical averages) like internal energy, specific heat, etc., of the system.

For example, in nuclear physics, the experimental graph of the excitation spectra of any heavy nuclei (plot of energy vs transition strengths) typically shows peaks and
widths. The peaks give the energy levels, the height of the peak determines the transition strength and the width determines the neutron decay width. There is such a large number of levels that it is more useful to study the statistics of the spacing between successive levels than to list the individual values, which show no apparent structure apart from a reluctance to lie too close to one another. This is why Wigner wondered if they could be understood as the eigenvalues of a matrix with random entries governed by some probability distribution. We strongly urge the readers to see the pedagogical article on universal properties [1] which includes a figure for the energy levels of Gadolinium-156 and their connections to the theory of random matrices.

Wigner realised, while trying to understand the experimental data of the slow neutron resonance spectra of many heavy nuclei, that the Hamiltonian $H$ has to be taken as a matrix with random matrix elements. He proposed that the local statistical behaviour of the energy levels of heavy nuclei are similar to the distribution of the eigenvalues of a random matrix. In fact, he tried to fit an empirical formula for the probability distribution function $p(s)$ of the spacing $s$ between any two neighbouring energy levels which will reproduce the experimental results. He also derived the energy level density $\sigma(E)$ (number of levels in the unit energy interval around energy $E$).

So, the key idea here is to look at the Hamiltonian as a member of an ensemble of random matrices (collection of Hamiltonians). These ensembles are usually called matrix models. Recently, the theory of random matrices has found applications in diverse fields like biology, telecommunications, financial stock market, etc.

Keywords
Matrix models, Feynman diagrams.
The structure of this article is as follows: In Section 2, we briefly present the integral calculus of matrices. In Section 3, we present Feynman’s approach of systematically studying the theory of random matrices. In particular, we elaborate on the algebraic and the diagrammatic method for a system with cubic interaction. In Section 4, we present a few applications of integral calculus of random matrices. In Section 5, we indicate computer programmes which are useful for the integral calculus of random matrices.

2. Integral Calculus of Matrices

We are all familiar with multiple integration of functions $f(x_1, x_2, \ldots x_N)$, where $N$ denotes the number of integrations and $x_i$ are the integration variables. Suppose we replace such functions by $f(\{M_{ij}\})$, where $M_{ij}$ with $i, j \in (1, N)$ denote the matrix elements of $N \times N$ matrix. Matrix models deals with a systematic procedure to do the integration of the $N^2$ matrix elements of $f(\{M_{ij}\})$. In the next section, we will give a flavour of Feynman’s algebraic and diagrammatic approach to formally study integral calculus of $N \times N$ matrices with random matrix elements. In physics, one can treat these matrices as representing Hamiltonians and we are essentially trying to obtain average properties of the system described by such Hamiltonians.

In the kinetic theory of gases, for finding mean speed $\bar{v}$ or root mean square speed $v_{\text{rms}}$ of the gas, we integrate over the three components of the velocity using Boltzmann distribution function $P(E) = \exp(-\frac{E}{kT})$, where $E$ is the energy, $T$ is the temperature and $k$ is the Boltzmann constant. $P(E)dE$ denotes the probability of the system to be found in the energy interval $E$ and $E+dE$. Following the same approach, we can study the average values
of functions of the elements of a random matrix using a Gaussian distribution function of matrices: $P_G(M) = \exp(\text{tr}(M^2)/2)$. We can go beyond the Gaussian distribution to study many complex systems where the distribution function involves traces of higher powers of matrices: $P(M) \propto \exp(-\frac{1}{2}\text{tr}M^2 - g_3\text{tr}M^3 + g_4\text{tr}M^4 \ldots )$, where $g_3, g_4, \ldots$ are real constants. We shall now briefly discuss the integral calculus for the ensemble of matrices described by the Gaussian distribution function.

2.1 Gaussian System

Suppose we consider random $N \times N$ real symmetric matrices governed by a Gaussian probability distribution. There are $N(N + 1)/2$ random variables $M_{ij}$ ($i \leq j \in (1, N)$). Under an orthogonal transformation $O$, the random matrices transform as $M \rightarrow M' = O^T MO$. The probability of finding a $N \times N$ matrix in a given ‘volume element’ $dM = \prod_i dM_{ii} \prod_{j < i} dM_{ij}$ is given by

$$P(M)dM = \prod_i dM_{ii} \prod_{j < i} dM_{ij} \exp(-\frac{1}{2}\text{tr}M^2). \quad (1)$$

To determine the probability distribution function for a specific set of eigenvalues $x_1, x_2, \ldots x_N$, we can do an orthogonal transformation and rewrite the volume element

$$dM = \det\left(\frac{\partial M_{ij}}{\partial x_i} \quad \frac{\partial M_{jk}}{\partial x_i}\right) dx_1 dx_2 \ldots dx_N dp_1 \ldots dp_{\ell}, \quad (2)$$

where $p_i$’s with $i \in (1, \ell = N(N - 1)/2)$ are the independent variables of the orthogonal matrix $O$. The determinant is simply the Jacobian in transforming from one set of variables $M_{ij}$’s to another set of variables $x_i$’s, $p_j$’s.
For definiteness and clarity, let us do the above calculation of Jacobian for the lowest non-trivial $2 \times 2$ real symmetric matrix $M$. In this case, there are 3 independent variables. The $2 \times 2$ orthogonal matrix $O$ which has one independent variable diagonalises the matrix $M$. That is,

$$
\begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix} =
\begin{pmatrix}
O_{11} & O_{12} \\
O_{21} & O_{22}
\end{pmatrix}
\begin{pmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{pmatrix}
\begin{pmatrix}
O_{11} & O_{21} \\
O_{12} & O_{22}
\end{pmatrix},
\end{equation}

We take the orthogonal matrix elements to be functions of $p$, i.e., $O_{ij}(p)$, and $\lambda_1$ and $\lambda_2$ to be eigenvalues of the matrix $M$.

The Jacobian for this case is the determinant of the matrix:

$$
K = \begin{vmatrix}
\frac{\partial M_{11}}{\partial \lambda_1} & \frac{\partial M_{12}}{\partial \lambda_1} & \frac{\partial M_{12}}{\partial \lambda_2} \\
\frac{\partial M_{11}}{\partial \lambda_2} & \frac{\partial M_{22}}{\partial \lambda_1} & \frac{\partial M_{12}}{\partial \lambda_2} \\
\frac{\partial M_{11}}{\partial p} & \frac{\partial M_{22}}{\partial p} & \frac{\partial M_{12}}{\partial p}
\end{vmatrix},
\end{equation}

Substituting matrix elements $M_{ij}$ in terms of the eigenvalues and the $O_{ij}$'s, the determinant of the above matrix will be

$$
J = \det K = f(p)(\lambda_1 - \lambda_2),
\end{equation}

where $f(p)$ contains matrix elements of $O$ and its derivatives. Thus, we see that the Jacobian is proportional to the difference of the eigenvalues and the proportionality constant is dependent on $p$. The property of the orthogonal matrix $O$, $OO^T = O^TO = I$ has been used to get the final simplified expression for the Jacobian.

The method elaborated for $2 \times 2$ real symmetric matrices can be generalised to obtain the Jacobian for all $N \times N$ real symmetric matrices. In fact, one can write a small programme using Mathematica to evaluate the Jacobian for $2 \times 2$ matrix, $3 \times 3$ matrix, etc., which we
leave as an exercise for the reader. The final result for the probability distribution for a specific set of eigenvalues, after inserting the Jacobian matrix and integrating the $p_i$ variables, is

$$P_{N\beta}(x_1, x_2, \ldots, x_N) = C_{N\beta} \exp \left( -\frac{1}{2\beta} \sum_{i=1}^{N} x_i^2 \right) \prod_{k<j} |x_j - x_k|^\beta,$$  \hspace{1cm} (4)

where $\beta = 1$ for real symmetric matrices.

The same exercise can be carried out for a system described by Hermitean matrices with random matrix elements. In this case, the value of $\beta$ will turn out to be 2. The constant factor $C_{N\beta}$ is simply a normalization constant that ensures that the total probability is one. The parameter $\beta$ is introduced mainly to see an analogy with a certain one-dimensional gas of $N$ interacting particles, that we now describe.

Consider the gas to be made of $N$ point particles having positions $x_1, x_2, \ldots, x_N$, subject to a potential

$$W(\{x_i\}) = \frac{1}{2} \sum_i x_i^2 - \sum_{i<j} \log |x_i - x_j|. \hspace{1cm} (5)$$

The probability of finding the system in a state with given $W$ at temperature $T$ is

$$P = C \exp \left( -\frac{W(\{x_i\})}{kT} \right) = C \exp \left( -\frac{1}{2kT} \sum_i x_i^2 \right) \prod_{k<j} |x_k - x_j|^{1/kT} \hspace{1cm} (6)$$

which has the same form as (4) with $\beta = 1/kT$.

In this section we have concentrated on the probability distribution function for a set of eigenvalues in a Gaussian ensemble. Our goal is to systematically study the
theory of $N \times N$ random matrices using Feynman’s approach which will be useful to study average properties or correlations in any complex system. We shall elaborate Feynman’s method for Hermitean matrices in the following section.

3. Feynman Method

There are two ways of solving the theory of random matrices. The simplest but tedious method is to solve algebraically. The other elegant method is a diagrammatic method. We will present both the methods for matrices with finite dimension $N$.

In the Feynman method, for a system described by the distribution function $P(M)$, we define a partition function $Z$ in the following way:

$$Z = \frac{1}{C} \int P(M)dM,$$

where the matrices $M$’s are $N \times N$ Hermitean matrices and $C$ is a constant. We introduce an action $S(M)$ which is related to the distribution function $P(M)$ as follows:

$$P(M) = \exp(-S(M)).$$

Once we have the distribution function, we can define any correlation function or averages of an arbitrary function of the matrix elements. The average of a function $f(M)$ is given by

$$\langle f \rangle = \frac{1}{C} \int f(M)e^{-S(M)}dM.$$

We consider the form

$$S(M) = \frac{1}{2g_s}\text{tr}M^2 + \frac{g_3}{3g_s}\text{tr}M^3 + \frac{g_4}{4g_s}\text{tr}M^4 + \ldots$$

Here, $g_s$ and the cubic, quartic and higher coupling strengths $g_3, g_4, \ldots$, etc., are constants characteristic of the system.
For clarity, we will first look at the computation of partition function for a complex system with a cubic term. That is,

\[ Z = \frac{1}{C} \int \exp \left( -\frac{1}{2g_s} \text{tr}M^2 - \frac{g_3}{3g_s} \text{tr}M^3 \right) \, dM. \quad (11) \]

This can be expanded in powers of \( g_3 \) as follows:

\[ Z = \frac{1}{C} \int \exp \left( -\frac{1}{2g_s} \text{tr}M^2 \right) \left[ 1 - \frac{g_3}{3g_s} \text{tr}M^3 \right. \\
+ \left. \left( \frac{g_3}{3g_s} \right)^2 (\text{tr}M^3)^2 + \ldots \right] \, dM. \quad (12) \]

We will describe the algebraic way of determining the terms in this expansion.

### 3.1 Algebraic Method

The zeroth order term (term independent of \( g_3 \)) in the above expansion is a Gaussian ensemble partition function which we will denote as \( Z_G \):

\[ Z^{(0)} = \frac{1}{C} \int \exp \left( -\frac{1}{2g_s} \text{tr}M^2 \right) = Z_G. \quad (13) \]

The first order term (term with one power of \( g_3 \)) is

\[ Z^{(1)} = \frac{1}{C} \int \exp \left( -\frac{1}{2g_s} \text{tr}M^2 \right) \frac{g_3}{3g_s} \text{tr}M^3. \quad (14) \]

The following mathematical trick is useful to solve the above matrix-integrals:

\[ I = \int_{-\infty}^{\infty} e^{-x^2/\pi} x^m \, dx = \left[ \frac{\partial^m}{\partial h^m} \int_{-\infty}^{\infty} e^{-x^2/\pi + h x} \, dx \right]_{h=0}. \quad (15) \]

We can complete the square and rewrite the integral as

\[ I = \left[ \frac{\partial^m}{\partial h^m} \left( e^{h x^2/2} \int_{-\infty}^{\infty} e^{-\frac{(x-h)^2}{2}} \, dx \right) \right]_{h=0} \\
= \left( \int_{-\infty}^{\infty} e^{-\frac{(x-h)^2}{2}} \, dx \right) \left[ \frac{\partial^m}{\partial h^m} \left( e^{h x^2/2} \right) \right]_{h=0}. \quad (16) \]
Expanding $\text{tr}M^3$ in terms of the matrix elements, equation (14) becomes

$$Z^{(1)} = \frac{1}{C} \frac{g_3}{3g_s} \int \text{d}M e^{-\frac{1}{2g_s} \text{tr}M^2} (M_{ab} M_{bc} M_{ca}), \quad (17)$$

Here ‘repeated indices’ implies that they are summed over. Now we perform a similar trick (15,16) involving matrices by introducing a term $\text{tr}(JM)$ in the exponent and completing the square. The role of $h$ in the mathematical trick is played by matrix $J$ to solve higher order terms in the partition function as follows:

$$Z^{(1)} = \frac{1}{C} \frac{g_3}{3g_s} g_s^3 \left( \frac{\partial}{\partial J_{ba}} \frac{\partial}{\partial J_{cb}} \frac{\partial}{\partial J_{ac}} \right) \times \int \text{d}M e^{-\frac{1}{2g_s} \text{tr}(M-J)^2} e^{\frac{1}{2g_s} \text{tr}J^2} \big|_{J=0}
= \frac{1}{C} \frac{g_3}{3g_s} g_s^3 Z_G \left( \frac{\partial}{\partial J_{ba}} \frac{\partial}{\partial J_{cb}} \frac{\partial}{\partial J_{ac}} \right) \times e^{\frac{1}{2g_s} \text{tr}J^2} \big|_{J=0}. \quad (18)$$

Clearly, the first derivative pulls out a factor of $J$ from the exponent and the second derivative acts on it or introduces another $J$. Hence, the net result of the three derivatives is to introduce a factor of $J$ or $J^3$, which in the limit $J = 0$, becomes zero. Hence there is no contribution from the first order term in the cubic interaction. Therefore, we need to look at the second order term:

$$Z^{(2)} = \frac{1}{2!C} \left( \frac{g_3}{3g_s} \right)^2 \int \text{d}M e^{-\frac{1}{2g_s} \text{tr}M^2} (\text{tr}M^3)^2. \quad (19)$$

Rewriting in terms of matrix elements, we get

$$Z^{(2)} = \frac{1}{2!C} \left( \frac{g_3}{3g_s} \right)^2 \times \int \text{d}M e^{-\frac{1}{2g_s} \text{tr}M^2} (M_{ab} M_{bc} M_{ca})(M_{de} M_{ef} M_{fd}). \quad (20)$$
Introducing $J$ and doing the mathematical trick, the second order term is

$$
Z^{(2)} = \frac{1}{2C} \left( \frac{g_3}{3g_s} \right)^2 \frac{g_s^6 Z_G}{2} \left( \frac{\partial}{\partial J_{ba}} \frac{\partial}{\partial J_{ch}} \frac{\partial}{\partial J_{ac}} \right) \times \left( \frac{\partial}{\partial J_{cd}} \frac{\partial}{\partial J_{fe}} \frac{\partial}{\partial J_{gf}} \right) e^{\frac{1}{2g_s} \text{tr} J^2} \bigg|_{J=0}
$$

$$
= \frac{1}{C} \left( \frac{g_3}{3g_s} \right)^2 \frac{g_s^2 Z_G}{2} \sum (3\delta_{ae}\delta_{cf}\delta_{bd} + 3\delta_{ae}\delta_{bd}\delta_{cf}\delta_{ce}\delta_{af} + 9\delta_{bc}\delta_{ae}\delta_{cf}\delta_{de}\delta_{ef}) . \tag{21}
$$

The summation symbol implies that all the indices $a, b, c, d, e, f$ are summed over. These indices assume values from 1 to $N$. Let us elaborate the summation of the first term in parenthesis involving three Kronecker deltas: Here, $a$ can assume $N$ values, but once $a$ is fixed, $e$ is fixed because of the Kronecker delta. Similarly, one can choose $c$ and $b$; and that choice fixes $f$ and $d$. Thus the net contribution from this term is $3N^3$. Similarly, the second term contributes $3N$ and the third term gives $9N^3$. The final result is:

$$
Z^{(2)} = \frac{1}{2C} \left( \frac{g_3}{3g_s} \right)^2 \frac{g_s^2 Z_G}{2} \left(12N^3 + 3N\right) . \tag{22}
$$

The higher order terms in the partition function can be similarly computed using the mathematical trick.

We can also find the correlation functions (9) associated with the complex sytem with the cubic interaction. For example, the two-point correlator $\langle M_{ij}M_{kl} \rangle$ at zeroth order will be

$$
\langle M_{ij}M_{kl} \rangle^{(0)} = \frac{1}{C} \int dM \exp \left( -\frac{1}{2g_s} \text{tr} M^2 \right) M_{ij}M_{kl}
$$

$$
= \frac{Z_G g_s^2}{C} \left( \frac{\partial}{\partial J_{ii}} \frac{\partial}{\partial J_{kk}} \right) e^{\frac{1}{2g_s} \text{tr} J^2} \bigg|_{J=0}
$$

$$
= \frac{Z_G}{C} g_s \delta_{ii} \delta_{jk} . \tag{23}
$$
The algebraic method becomes tedious at higher orders but definitely computable. Feynman introduced a neater diagrammatic method of understanding these algebraic expressions which gives an elegant way of obtaining higher order terms.

The first order contribution to the two-point function is

$$\langle M_{ij}M_{kl}\rangle^{(1)} = \frac{1}{C} \int dM \exp \left( -\frac{1}{2g_s} \text{tr} M^2 \right) \left( \frac{g_3}{3g_s} \text{tr} M^3 \right) \times M_{ij}M_{kl},$$

which will be zero when we apply the trick (15,16) involving $J$ matrix. Following this procedure, we can compute higher order contributions to the two-point function as well. Though we have elaborated on the partition function and correlators for the cubic interaction, the method is general enough to study any complex system described by action (10). The algebraic method becomes tedious at higher orders but definitely computable. Feynman introduced a neater diagrammatic method of understanding these algebraic expressions which gives an elegant way of obtaining higher order terms.

### 3.2 Diagrammatic Method

The fundamental diagrams which are useful to study any higher order term in the partition function and correlator functions are:

- The matrix element $M_{ij}$ is represented by two anti-parallel lines with indices $i$ and $j$ as shown in Figure 1.

- The cubic interaction term $\text{tr} M^3 = M_{ij}M_{jk}M_{ki}$ is shown in Figure 2. Notice that the orientations in

![Figure 1. Representation of matrix element $M_{ij}$](image1)

![Figure 2. Three-point vertex.](image2)
the antiparallel lines are matched. This diagram is usually called three-point vertex to indicate that there are three pairs of antiparallel lines intersecting.

- The diagrams representing other interactions like \( \text{tr} M^n \) will be an \( n \)-point vertex diagram.

We will now use the following set of rules to represent any higher order contribution to the partition function describing cubic interaction:

1. For the \( m \)-th order contribution, we introduce \( m \) three-point vertices. So, for the second order contribution, we have two three-point vertices.

2. Then, we join all the antiparallel lines so that the diagram looks like a complete graph. These graphs are called fatgraphs. The fatgraph representing a second order contribution is shown in Figure 3.

3. For every fatgraph, we can count the number of loops (which is also the number of faces \( F \)), the number of three-point vertices \( V \) and the number of antiparallel line pairs connecting the vertices (also denoted as number of edges \( E \)). For Figure 3, \( F = 3 \), \( V = 2 \) and \( E = 3 \).

Figure 3. A second order term.
4. There are a number of ways of joining the vertices by antiparallel line pairs which gives the same fatgraph. For example, Figure 3 can be obtained in three ways. So, we associate a combinatorial factor $r$ for every fatgraph.

5. Given a fatgraph, we say that the contribution to the partition function is $(Z_G/C)r N^F g^E_s (g_3/g_s)^V$. For Figure 3, it is $(Z_G/C)3N^3 g_s g_3^2$ which is matching with the first term in (21).

Following the above set of rules, we can see that it is impossible to draw any fatgraph with odd number of three-point vertices. Hence, we can confidently say that any odd order contribution to the partition function is zero without working out the tedious algebra (18). The other two terms in (21) can be drawn as fatgraphs as shown in Figures 4 and 5 respectively.

We see that $F = 1, V = 2, E = 3$ characterises Figure 4. Also, it is easy to see that there are again only three ways of drawing this fatgraph. So, this diagram contributes $(Z_G/C)3N g_s g_3^2$. Similarly, Figure 5 has $F = 3, V = 2, E = 3$ and there are nine ways of obtaining the fatgraph. Hence the diagram contributes $(Z_G/C)9N^3 g_s g_3^2$. Thus, we have an elegant way of determining any higher order contribution to the partition function.
To study $n$-point correlation functions diagrammatically, we can again follow the above set of rules but the graphs must have $n$ external antiparallel line pairs which are not connected. Following this rule, we represent a two-point function in Figure 6 which involves a region coloured blue with two external antiparallel line pairs.

We can place the fundamental diagrams (Figures 1 and 2) inside the blue region so that all their antiparallel line pairs are connected to each other or to the external antiparallel line pairs.

For example, the zeroth order contribution to two-point function will involve only Figure 1 inside the shaded region and connected to the two external antiparallel line pairs. So, the matching of indices of the two external antiparallel line pairs by an internal antiparallel line pair gives the two Kronecker deltas and a factor of $g_s Z_G/C$ (following the set of rules) agreeing with (23). In the same way, by placing a three-point vertex (Figure 3) in the blue region, there is one antiparallel line pair which is unconnected. This is equivalent to saying that the first order contribution to two-point function is zero which confirms the algebraic result (24). All even order terms will require even number of three-point vertices inside the blue region. It is not difficult to see that the antiparallel line pairs inside the blue region are all connected to each other or with the external two antiparallel lines. Hence the even order terms will give non-zero contribution to the two-point function and they can be directly
written down after drawing the graph and following the set of rules.

Even though we have elaborated for the cubic interaction, it is not difficult to extrapolate the rules for any interaction as follows: We associate a factor of \( N \) with every face, a factor \( g_p/g_s \) with each \( p \)-point vertex and a factor of \( g_s \) with every edge. Thus, a fatgraph with \( V_p \) number of \( p \)-point vertices will contribute

\[
\prod_p \left( \frac{g_p}{g_s} \right)^{V_p} N^F g_s^{E-V} = N^F g_s^{E-V} \prod_p g_p^{V_p}, \tag{25}
\]

where \( V = \sum_p V_p \) gives the total number of vertices. The contribution from each fatgraph is then proportional to \( N^F g_s^{E-V} \). In topology, the number of faces \( F \), vertices \( V \) and edges \( E \) of a polyhedron are related to its genus \( g \) by

\[
2 - 2g = F + V - E. \tag{26}
\]

Crudely speaking, the genus of a surface is the number of handles on it. A sphere has genus 0 and a torus has genus 1.

Introducing the parameter \( t = Ng_s \), the contribution (25) is found to be proportional to \( t^{E-V} N^{2-2g} \). Hence the partition function from the diagrammatic method can be formally written as

\[
Z = \exp(F) = \exp \left( \sum_g F_g(t, \{g_i\}) N^{2-2g} \right), \tag{27}
\]

where \( F \) is referred to as free energy and \( F_g(t, \{g_i\}) \) is the genus \( g \) contribution to the free-energy which depends on \( t \) and the coupling constants \( g_3, g_4, \ldots \).

Suppose we take the following limit, \( N \to \infty \) and \( g_s \to 0 \) such that \( t \) is fixed, we see that the contribution from
\( g = 0 \) will be dominant. So, to get the genus 0 contribution, we need to draw all fatgraphs with \( V - E + F = 2 \) which are usually called as planar graphs.

We have drawn all possible fatgraphs at second order. Only Figures 3 and 5 satisfy the condition \( V - E + F = 2 \) and hence planar. So, we need to draw all planar graphs at every order and sum all the contributions to determine \( F_0(t, \{g_i\}) \). Clearly, doing such an infinite sum is not an easy task.

When the dimensions of matrices are large \( (N \to \infty) \), we can directly determine the genus 0 answer (contributions from all planar diagrams) using another approach. This method involves complex analysis called \textit{saddle point analysis} which we shall discuss in a future article.

The method elaborated for a single matrix \( M \) with \( N^2 \) random matrix elements can be generalised for multi-matrix models involving many matrices. Also, these matrix elements which are independent of space and time coordinates can be generalised to matrices with matrix elements dependent on space and time coordinates. This leads to a vast subject called matrix field theory. We briefly present the applications of matrix models and matrix field theory in diverse fields.

\section*{4. Applications}

\subsection*{4.1 Physics}

One of the questions we could ask in the theory of random matrices is the following: Given an ensemble of real symmetric \( N \times N \) random matrices \( (M) \), what is the probability of finding a specific set of \( n \leq N \) eigenvalues? Suppose, we choose two of the eigenvalues as \( x_1 = -s/2 \) and \( x_2 = s/2 \) in (4), then the probability for these two
eigenvalues gives Wigner’s spacing distribution \( p(s) \):

\[
p(s) \approx \int \ldots \int_{\text{out}} P_{N_1}(x_1 = -s/2, x_2 = s/2, \ldots x_N) \, dx_3 \ldots dx_N , \tag{28}
\]

where the \textit{out} integration implies that \( |x_i| > s/2 \) for \( i \in (3,N) \). Similarly, the energy density function \( \sigma(x) \) can be obtained by choosing one eigenvalue \( x_1 = x \) and others integrated as follows:

\[
\sigma(x) = N \int P_{N_1}(x, x_2, \ldots x_N) \, dx_2 dx_3 \ldots dx_N . \tag{29}
\]

Wigner tried to derive \( \sigma(x) \), in the limit of large \( N \), using the gas analogy. If \( N \) is large, the system of particles may be treated as a continuum such as a classical fluid with a continuous density function \( \sigma(x) \):

\[
\sigma(x) = \lim_{N \to \infty} \sum_{i} \delta(x - x_i) \to \int \delta(x - x') dx' , \tag{30}
\]

where the summation over eigenvalues \( x_i \)'s can be replaced by integration when \( N \) is large. The function is subject to the condition

\[
\int_{-\infty}^{\infty} \sigma(x) dx = N . \tag{31}
\]

Then the potential (5), in the continuum, can be taken as

\[
W(\sigma) = \frac{1}{2} \int_{-\infty}^{\infty} x^2 \sigma(x) dx - \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(x) \sigma(y) \log |x - y| dx dy . \tag{32}
\]

From the hypothesis of statistical mechanics, the system will be at equilibrium in the state for which \( W \) is minimum. The form of \( \sigma \) that minimizes \( W \), after solving the integral equation, turns out to be

\[
\sigma(x) = \begin{cases} 
\frac{1}{\pi} (2N - x^2)^{1/2} & |x| \leq (2N)^{1/2} \\
0 & |x| > (2N)^{1/2}
\end{cases} . \tag{33}
\]
This function is Wigner’s famous semi-circle law. Thus we see that Wigner’s spacing distribution and the density function are reproduced by the ensemble of real symmetric matrices with Gaussian distribution function. We strongly urge the readers to take a look at the book by Mehta [2] where the applications and formalism of random matrices are presented.

4.2 Mathematics

Instead of real symmetric matrices, we could have considered an ensemble of Hermitean matrices with Gaussian distribution function. Surprisingly, the eigenvalue distribution of this ensemble is similar to the distribution of nontrivial zeros of Riemann zeta function conjectured to be on the line \( \text{Re}(z) = 1/2 \).

4.3 Biology

One of the challenging problems in molecular biology is RNA folding. Take an RNA chain of length \( L \) made of a sequence \( s_1s_2 \ldots s_L \), where each \( s_i \) takes one of the four possible letters C, G, A, U (the letters denote the nucleotides cytosine, guanine, adenine and uracil). This sequence is typically called the primary structure. Amongst the nucleotides, there is attraction between C and G and between A and U nucleotides. Such attraction between the nucleotides folds the RNA chain. Biologists are keen on determining the ground state configuration of the RNA-folded structure from the partition function at low temperatures. The tools of matrix models is helpful to study this problem. We refer the interested reader to [3] which elaborates on the matrix field theory approach for RNA folding.

4.4 Finance

An interesting problem in finance is to obtain the best
An interesting problem in finance is to obtain the best return from a set of financial assets, characterized by their average return and their risk. This is called portfolio optimisation. The theory of random matrices has been used to show that there are exponentially large number of optimal solutions. We leave the interested reader to see [4] for details.

5. Computer Programming

Some of the basic results given in this article and in [2] can be tested numerically by writing a small code with the help of softwares such as Mathematica and MATLAB or programming in FORTRAN or C/C++. Various random number generators are available online for use with FORTRAN, while C++, Mathematica and MATLAB have inbuilt random number generators. We can use the random number generator to generate $2 \times 2$ random matrices and obtain a distribution function $p(x_1, x_2)$ which should agree with (4). This can be indirectly checked by summing one of the eigenvalues $x_2$ to obtain $\sigma(x_1)$ (29) and verifying (31). We expect the distribution of eigenvalues for $N \times N$ matrices for large $N$ to obey Wigner’s semicircle law. For a real symmetric $5000 \times 5000$ matrix with random elements, one can plot $\sigma(x)$ vs $x$ using the Mathematica program [5] which matches well with Wigner’s semi-circle law (33).

6. Conclusions

In this article, we have studied integral calculus of matrices. We have elaborated in detail on the algebraic and diagrammatic approach of Feynman for systems with cubic interaction. We have also shown that the planar diagrams give the dominant contribution when the dimensions of the matrices ($N \to \infty$) are large. We have mentioned the applications of the theory of random matrices in diverse fields. Finally, we have indicated the
usefulness of random number generators in the programming packages for obtaining Gaussian distribution function and density function.

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Suggested Reading


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