Fermat: A Versatile Genius

Utpal Mukhopadhyay

Fermat, a lawyer by profession, was a mathematical genius. Apart from being regarded as the founder of modern number theory, he made seminal contributions to calculus and probability theory. In this article, various mathematical works of Fermat are discussed briefly along with the notoriously difficult FLT associated with him.

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Pierre de Fermat (1601–1665) was one of three contemporaries who were the forerunners of significant mathematical ideas that revolutionized mathematics, the other two being René Descartes (1596–1650) and Blaise Pascal (1623–1662). Descartes and Fermat formulated the basic ideas of calculus, Fermat excelled in the theory of numbers, and the foundations of probability theory were laid by Fermat and Pascal together. Fermat was an amateur mathematician, but was at par with leading mathematicians of the world. Moreover, he posed a famous mathematical problem which, in spite of being attacked by several mathematicians of high repute, remained unconquered till the last decade of the twentieth century.

We present below some glimpses of the multi-faceted mathematical talent of Fermat.

1. Brief Life Sketch

Fermat was born on August 20, 1601 in Beaumont-de-Lomagne in south-east France. His father was a trader of leather and his mother's family consisted of government lawyers. Fermat spent most of his life in Toulouse (France). By profession he was a jurist in the administration of French Emperor Louis XIV. Apart from performing his official duties efficiently, he had enough

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enthusiasm to exercise his intellect in other areas as well. He had a sound knowledge of major European languages, and his poems, written in Latin, French and Spanish, were well received. But he is best remembered for his outstanding mathematical talent. According to the customs of the time in France, to maintain the impartiality of judges, they were not allowed to mix freely with common people. So, each evening, as a pastime, Fermat exercised his brain by thinking over mathematical problems. In that respect he was an amateur mathematician. But he did seminal work in various branches of mathematics, viz., calculus, theory of probability, coordinate geometry, and above all, number theory. He used to exchange letters with contemporary mathematical stalwarts like Descartes and Pascal. Fermat breathed his last in 1665.

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We now present some aspects of Fermat's mathematical work.

2. Calculus

Fermat made significant contributions in both differential and integral calculus. In 1637, as an extension of his work on locus, he wrote a paper entitled *Methodus ad disquirendam maximum et minimum* (Method of Finding Maxima and Minima). This work was published posthumously. According to Fermat, the value of a continuous function changes very slowly near a maximum or minimum value. As early as 1615, Johannes Kepler (1571–1630) had had the same idea in mind when he said, "Near a maximum the decreaments on both sides are in the beginning only imperceptible" [1].

Fermat made significant contributions in both differential and integral calculus.

Fermat devised a method for dividing a line segment into two parts such that the area of the rectangle contained by the parts is a maximum. His method can be described as follows. Suppose the line segment AB has to be divided at C such that AC · CB is a maximum (*Figure* 1). Let AB $= a$, AC $= x$ and AC$' = x + e$,

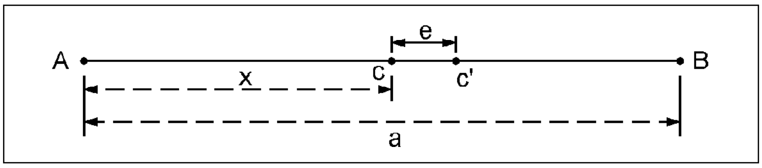

Figure 1.

where CC′ ($= e$) is so small that AC · CB and AC′· C′B can be regarded as nearly equal, i.e., AC · CB ≈ AC′ · C′B.

Therefore, $x(a-x) \approx (x+e)(a-x-e)$, and so $a-2x-e \approx 0$. When $e \approx 0$, $x \approx a/2$; this means that the value of AC · CB will be maximum if AB is bisected at C, which implies that the rectangle must be a square.

Fermat also discovered a method for drawing a tangent to a curve at any arbitrary point.

Fermat also discovered a method for drawing a tangent at any point on a curve. Suppose that P and P′ are two points on a curve close to each other (*Figure* 2). PT is the tangent to the curve at P and PN is the perpendicular from P upon the x-axis. Fermat's motivation was to find the length TN because by knowing TN, one can obtain the position of the point T. So, it will be possible to draw the tangent PT. He proceeded in the following manner.

In *Figure* 2, △ PQR and △ TPN are similar. So,

$$\text{TN}/\text{PN} = \text{PR}/\text{QR}. \quad (1)$$

Let TN $= t$, ON $= x$, PN $= y$, NN′ = PR $= e$ and P′R $= d$. Since P and P′ are close to each other, P′R ≈ QR. From (1) we get: $t/y \approx e/d$.

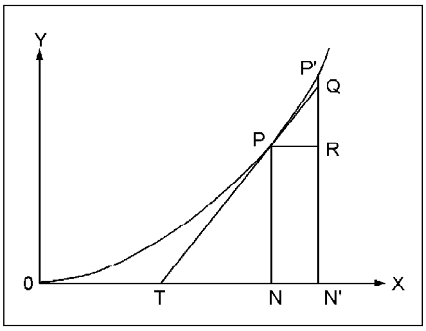

Figure 2.

Fermat used this method for different curves. For example, consider the curve $y = x^n$ where n is a positive integer. Let $\mathrm{P} = (x, y)$ and $\mathrm{P}' = (x+e, y+d)$ respectively. Then $y + d = (x+e)^n = x^n + nx^{n-1}e +$ terms involving higher powers of e. Hence $x^n + d = x^n + nx^{n-1}e +$ higher powers of e. So $d/e = nx^{n-1} +$ terms involving higher powers of e. Now,

$$t = \frac{ye}{d} = \frac{x^n}{nx^{n-1} + \text{terms in involving higher powers of } e}.$$

Putting $e = 0$, we get $t = x/n$. Again,

$$\text{Slope of the tangent at P} = \frac{\mathrm{PN}}{\mathrm{TN}} = \frac{ny}{x} = \frac{nx^n}{x} = nx^{n-1}.$$

So the slope of the tangent at P is equal to the differential coefficient at that point. Fermat did not provide a formal proof for his method, but his technique is identical to the modern method for calculating the differential coefficient. For this reason, the famous French mathematician Pierre–Simon de Laplace (1749–1827) called Fermat 'the true inventor of differential calculus' [1]. Fermat made contributions in the field of integral calculus also. The method he adopted for finding the area of a plane region bounded by a curve is similar to the modern method.

Laplace called Fermat "the true inventor of differential calculus".

3. Number Theory

Fermat's favourite area of study was without doubt the theory of numbers. He discovered many properties of the integers, and hence he is considered today as 'The Father of Modern Number Theory'. Some of his major discoveries in this field are as follows:

- *If p is a prime number and a is any number coprime to p, then $a^{p-1} - 1$ is divisible by p* [2, 3]; in Gaussian notation $a^{p-1} \equiv 1 \pmod{p}$.

  This theorem, known as 'Fermat's theorem', was proposed by him in a letter in the year 1640 and

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subsequently proved by Euler in 1761. A corollary of this theorem states: "If p is a prime number and a is any number whatever, then $a^p - a$ is divisible by p" [2].

- *No prime number of the form* $4n + 3$ *can be expressed as a sum of two square numbers while every prime of the form* $4n + 1$ *has a unique representation as the sum of two squares.*
- *An odd prime number can be expressed as the difference of two squares in one and only one way*: $5 = 3^2 - 2^2$; $7 = 4^2 - 3^2$; $11 = 6^2 - 5^2$; etc.
- *The only rational solution of the Diophantine equation* $y^3 = x^2 + 2$ *is* $x = \pm 5$, $y = 3$.
- *The equation* $x^2 - Ay^2 = 1$, *where* A *is a given non-square positive integer, possesses an infinite number of integral solutions.* (This equation is known as the 'Pell equation' – but see *Box* 1 for some remarks on this name.)
- *If* $4n + 1$ *is a prime number, then* $(4n + 1)^s$ *is the hypotenuse of* s *distinct right-angled triangles.* For instance, when $n = 1$, we get the prime number 5.

Box 1. Pell's Equation

The name 'Pell equation' is a historical error. Euler wrongly attributed equations of the type $x^2 - Ay^2 = 1$ to John Pell (1610–1685), an English mathematician who incidentally was one of the doctoral supervisors of Christian Huygens (1629–1695). He mentioned the above equation in his book on algebra but did not solve it. The great Indian mathematician Brahmagupta (598–660) had solved this equation much earlier. But his solution had limitations and was subsequently improved upon by Bhaskaracharya II (1114–1185) using his *'cakravla'* (cyclic) method. So, the Pell equation should really be called the 'Brahmagupta–Bhaskara equation'.

It is interesting to note that Bhaskara II illustrated the solution of the equation $61x^2+1 = y^2$ in his book *Bijaganitam* by using cyclic method and Fermat showed the very same problem to Frenicle and other mathematician friends. Bhaskara II got the solution $x = 226153980$, $y = 1766319049$, but Fermat's friends failed to solve it. Many years later, Euler too solved it.

If $s = 2$, we get two triangles with hypotenuse 25, viz. $(25, 20, 15)$ and $(25, 24, 7)$. Fermat generalized the $3 : 4 : 5$ ratio of the sides of a right-angled triangle and for different values of s, at each step, a new triangle can be obtained whose sides are not in the ratio $3 : 4 : 5$.

- *If the integers a, b, c represent the sides of a right-angled triangle, then its area is not a square number.* This was proved later by Lagrange (1736–1813).

Fermat proved many of his theorems using a technique called "method of infinite descent". But he was more comfortable with negative propositions than affirmative ones.

It is surprising and curious that Fermat rarely gave detailed proofs of his theorems. In fact, he did not publish any research works related to number theory in his lifetime. Those results came to be known from his letters and his comments written in the margins of various books. His results were proved afterwards largely by Leonhard Euler (1707–1783) and Andre-Marie Legendre (1752–1833). His method of proof went into oblivion until a document entitled '*Relation des decouvertes en la science des nombres*' was recovered from the manuscripts of Huygens in 1879 in the library of Leyden [4]. From these we know that Fermat proved some of his theorems by using a technique known as '*la descente infinite ou indefinie*' (method of infinite descent). He used this technique for proving some negative propositions because he was more comfortable with negative statements than the affirmative ones. This is revealed in one of his letters where he stated: "For a long time I was unable to apply my method to affirmative propositions because the twist and the trick for getting at them is much more troublesome than that which I use for negative propositions" [1].

4. Quadrature

Fermat made significant contributions in the field of quadrature, i.e., the method of finding the area of a

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region bounded by a curve, the x-axis and two fixed ordinates. His was a pre-calculus method (calculus was yet to be discovered). He was particularly interested in the curve $y = x^n$. For $n = 2$, this curve reduces to the familiar parabola $y = x^2$ and hence the curve $y = x^n$ is known as a generalized parabola. For finding the area bounded by this curve, the x-axis and the ordinates $x = 0$ and $x = a$, Fermat divided the region into rectangles whose bases form a geometrical progression with common ratio r, where $0 < r < 1$ (*Figure* 3).

If the sum of the areas of the rectangles is A, then

$$A = a^n(a - ar) + (ar)^n(ar - ar^2) + (ar^2)^n(ar^2 - ar^3) + \cdots$$
$$= a^{n+1}\frac{1-r}{1-r^{n+1}}.$$

Fermat argued that as $r \to 1$, the area A tends to the area bounded by the curve. But he could readily recognize that for $r = 1$, A becomes undefined. So, he first factorized the denominator to cancel out $1 - r$ and arrived at the result

$$A = \frac{a^{n+1}}{1 + r + r^2 + \cdots + r^n}.$$

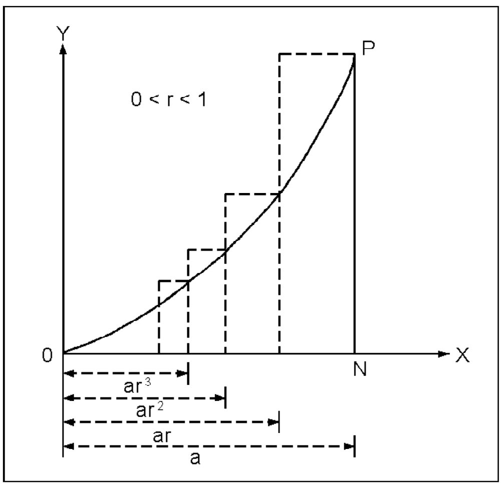

Figure 3.

Then he put $r = 1$ to obtain

$$A = \frac{a^{n+1}}{n+1}. \qquad (2)$$

Relation (2) is a familiar result obtained nowadays from the integral $\int_0^a x^n \, dx$. But, Fermat intuitively arrived at this result. Moreover, the above formula is valid not only for a particular curve but for a system of curves corresponding to different positive integral values of n. This is the importance of Fermat's work on quadrature.

This formula is valid not only for a particular curve but for a whole system of curves.

Fermat was successful in showing that the formula remains valid for negative integral values of n also, if the rectangles are drawn to the right of the line $x = a$ and the bases of the rectangles increase in the increasing x direction (*Figure* 4) but still form a geometrical progression with common ratio $r > 1$.Thus Fermat could generalize the result (2) though there is a huge difference between the curves $y = x^n$ when n is a positive integer and $y = x^m$ when m is a negative integer, because the former is everywhere continuous whereas the latter has a discontinuity at $x = 0$. (The second curve is known as a generalized hyperbola.)

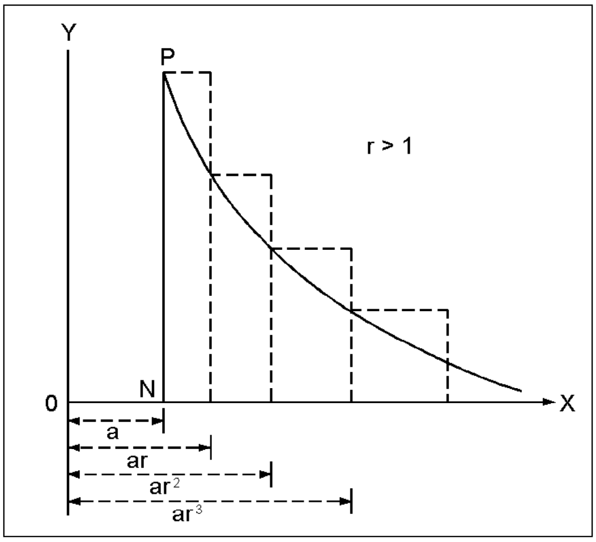

Figure 4.

However, formula (2) fails for $n = -1$. The curve now is a rectangular hyperbola $xy = 1$, where the coordinate axes are its two asymptotes. Fermat was aware of this difficulty and he commented: "I say that all these infinite hyperbolas except the one of Apollonius, or the first, may be squared by the method of geometric progression according to a uniform and general procedure" [1].

The troublesome case $n = -1$ was tackled by Grégoire (or Gregorious) de Saint-Vincent (1584–1667) who was a Belgian Jesuit and a contemporary of Fermat. He noticed that for $n = -1$, the rectangles have equal areas because in that case, the widths of the successive rectangles become $a(1 - r)$, $ar(1 - r)$, etc., and the corresponding heights are $1/a$, $1/ar$, $1/ar^2$ and so on. So, the area of each rectangle reduces to $(1 - r)$. As $r \to 1$, the areas become smaller and smaller and ultimately give us the required area.

5. Coordinate Geometry

Around the year 1629 Fermat began to edit the book *Plane Loci* of Apollonius (born *ca.* 262 BC). During that work he discovered the fundamental principle of coordinate geometry. According to Fermat's own version, this principle is: "Whenever in a final equation two unknown quantities are found, we have a locus, the extremity of one of these describing a line, straight or curved" [1]. Fermat discovered the principle at least a year prior to the publication of the famous *La Géométrie* of Descartes. Later, Fermat extended his work further and narrated his findings in the book *Ad locus planos et solidos isagoge* (Introduction to Loci Consisting of Straight Lines and Curves of the Second Degree). But that book was published in 1679, fourteen years after Fermat's death. Because of this circumstance Descartes is solely credited as the inventor of coordinate geometry. But by 1630, both Fermat and Descartes could express

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the equations of curves through algebraic equations and investigated their various properties. It may be mentioned here that the idea of locus was the central theme of Fermat's research on coordinate geometry.

A correspondence between Fermat and Pascal was instrumental in opening up a new branch of mathematics – probability theory.

6. Theory of Probability

A correspondence between Fermat and Blaise Pascal relating to a certain game of chance was instrumental in opening up a new branch of mathematics – the theory of probability. Chevalier de Mere proposed to Pascal the fundamental problem, to determine the probability which each player has, at any given stage of game, of winning the game. Pascal communicated this problem to Fermat, who solved it by the theory of combinations. The most important result enunciated by him states that if A has a chance p of winning a sum a, and a chance q of winning a sum b, then he may expect to win the sum $(ap + bq)/(p + q)$ [3].

7. Fermat's 'Least Time' Principle

In spite of the fact that Fermat was essentially a mathematician, physics problems also drew his attention. One of his major contributions to the world of physics is his 'principle of least time' which is related to the path followed by light rays. In fact, he developed Heron's idea of least path (see *Box* 2). Fermat described this principle in two letters written in the years 1657 and 1662. The original statement of Fermat's principle states: "The actual path between two points taken by a beam of light is the one which is traversed in least time" [5]. It may be formulated in a slightly different way in terms of optical path length as 'light, in going between two points, traverses the route having the smallest path length'. In the modern form, it can be stated in the following manner: 'A light ray, in going between two points, must traverse an optical path length which is stationary with respect to variation of the path.' This means that the path may be a maxima or minima or saddle point. The

One of Fermat's major contributions to the world of Physics is his 'principle of least time'.

Box 2. Heron's Problem

Heron was a Greek mathematician of Alexandria who lived in the first century AD In his book *Catroptica*, Heron posed the following problem. Suppose A and B are two given points lying on the same side of a line ℓ (*Figure* A). To find a point C on ℓ such that $|AC| + |CB|$ will be minimum.

Heron argued that if B_1 be the image of B under reflection in ℓ, then $|CB| = |CB_1|$, hence minimization of $|AC| + |CB|$ is equivalent to minimization of $|AC| + |CB_1|$. Now, the shortest path from A to B_1 is the straight line AB_1. So the point C will be the point of intersection of the line AB_1 with ℓ. Damianus (6th century AD), a commentator of Heron wrote [6]: "Heron ...showed that lines inclined at equal angles are the smallest of all intermediate ones inclined on the same side of a single line. Proving this he says that if nature does not want a ray of light to meander to no purpose, then it breaks it at equal angles." We know that the point C has the property that $\angle\alpha = \angle\beta$. This is nothing but the law of reflection.

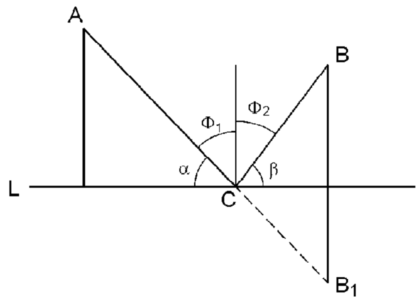

Figure A.

well-known laws of reflection and Snell's law of refraction of light can be derived by using Fermat's principle. In *Figure* 5, for a single reflection, the position of the

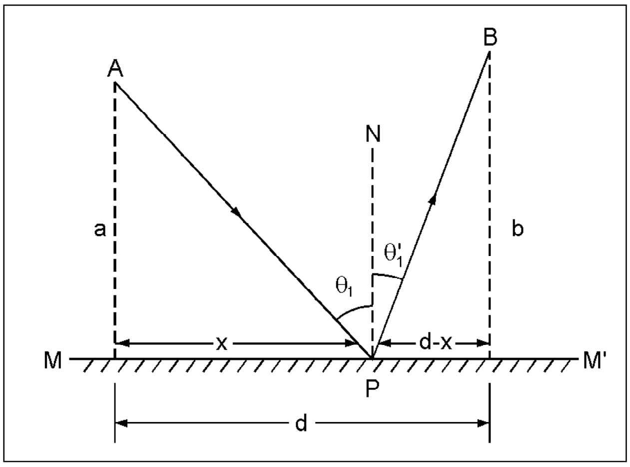

Figure 5.

point of incidence P will be such that the time taken by light to move from A to B via a reflection at P must be minimum (or a maximum or remain unchanged). If the total length AP + PB is denoted by s, then

$$s = \sqrt{a^2 + x^2} + \sqrt{b^2 + (d - x)^2},$$

where x is the distance of P from the foot of the perpendicular from A on the reflecting surface MM′. By calculating $\mathrm{d}s/\mathrm{d}t$ and equating it to zero it can be shown that $\angle$APN = $\angle$BPN, i.e., angle of incidence equals angle of reflection.

Later, Fermat's least time principle was generalized by Hamilton in his 'Principle of Least Action'.

Similarly, the laws of refraction can be deduced very easily. It may be mentioned here that laws of reflection and laws of refraction can be deduced from Huygens principle and Maxwell's equations as well. Afterwards, Fermat's principle was generalized by William Hamilton (1805–1865) in his 'principle of least action'.

8. Fermat's Last Theorem

Any writing on Fermat would be incomplete without mention of 'Fermat's Last Theorem' (FLT for short). So, let us end this article with a brief discussion on FLT.

It has already been mentioned that Fermat had the habit of writing his mathematical results and comments in the margins of books which he read. One of those comments gave birth to a notoriously difficult mathematical problem which became famous as 'Fermat's Last Theorem'. In the year 1621 Claude Bachet (1581–1638) published a Latin translation of the book *Arithmetica* written by Diophantus (210–290), a mathematician of Alexandria known as 'Father of Higher Arithmetic'. While reading a copy of that Latin translation in the year 1630, Fermat as usual wrote various comments in the margin of the book. The comment regarding problem number 8 made in page 87 of *Book* II of this Latin version goes as follows: "On the contrary, it is impossible to separate a cube into two cubes, a fourth power into two fourth

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Over time, all of Fermat's claims were proved, barring the FLT; that is how it got its name.

powers, or, generally any power above the second into powers of the same degree. I have discovered a truly marvelous demonstration of this proposition, which this margin is too narrow to contain." So, according to Fermat, the equation $a^n + b^n = c^n$ does not have any integral solution for $n > 2$. After Fermat's demise, his son Samuel published an edition of *Arithmetica* along with the comments of Fermat. After publication of that book, the comments of Fermat regarding his mathematical results created a sensation among mathematicians of the world. World famous mathematicians started proving the mathematical results of Fermat. Over time, all of Fermat's claims were proved, barring the one just mentioned. For this reason it came to be known as 'Fermat's Last Theorem'.

For $n < 3$, it is easy to find integral solutions of the above equation. For $n = 1$, the equation reduces to a triviality. For $n = 2$, the equation becomes $a^2 + b^2 = c^2$ which brings to mind the theorem of Pythagoras; we can find an infinite number of solutions because an infinite number of Pythagorean triplets of the forms $(3, 4, 5)$, $(5, 12, 13)$, $(8, 15, 17)$, $\ldots$ exist. But according to Fermat no such solutions exist for integers $n > 2$.

For nearly three and half centuries, FLT kept many renowned mathematicians on their toes. In spite of earnest efforts, it remained unproved. Progress was made for certain values of n. FLT was proved for $n = 3$ by Euler, while Fermat himself proved it for $n = 4$. In fact Fermat proved that the equation $a^4 + b^4 = c^2$ has no integral solution. This implies the FLT for $n = 4$, since the fourth power of a number is a perfect square. In the 1820s, French mathematician Legendre and German mathematician Peter Gustav Lejeune Dirichlet (1805–1859) proved the result for $n = 5$. Gabriel Lamé (1795–1870) of France proved it for $n = 7$, while Dirichlet proved it for $n = 14$. In 1847, German mathematician Ernst E Kummer (1810–1893) progressed a long way towards the general proof but ultimately failed. However, using Kummer's method it was possible to show that FLT is true for all integral values of $n < 100$. In the nineteenth century, lady mathematician Sophie Germain (1776–1831) made significant contributions in proving FLT. Germain showed that FLT is true for all prime numbers $n > 2$ for which $2n + 1$ is prime. But she failed to find a general proof, valid for all n. By mid-1993 it became possible, using computers, to show that FLT is true for all n less than four million.

In 1994, a Princeton mathematician Andrew Wiles created a worldwide sensation by proving FLT. Wiles had a dream since his childhood days that he would prove FLT. He proved it by establishing a special case of a far more general conjecture – the 'Taniyama–Shimura Conjecture', proposed by Japanese mathematician Yutaka Taniyama (1927–1958) in 1955 and given a more complete form by Taniyama's friend Goro Shimura (born 1930) of Princeton University. (The conjecture has been proved and is now called the *Modularity theorem.*) Wiles's paper 'Modular Forms, Elliptic Curves and Galois Representation' was published in the journal *Annals of Mathematics* in May 1995. It may be mentioned here that in 1908, a German industrialist Paul Wolfskehl had announced a prize of one lakh mark (two million dollars) for proving FLT; Wiles was eligible for this prize. In June 1997, in the Great Hall of Göttingen University in Germany, Wiles received the prize in the presence of five thousand mathematicians.

9. Epilogue

Fermat's Last Theorem is an excellent example of how a comment written in the margin of a book can generate much sensation and interest among mathematicians for so many years. After baffling both professionals and amateurs for nearly three and half centuries, the 'theorem' finally yielded to the tenacity and perseverance

To this day we do not know whether Fermat really had a proof of FLT.

of Andrew Wiles. The success of Wiles in proving FLT marked the end of a long quest indeed; many world famous mathematicians including Euler, Dirichlet, Legendre, etc., must have spent many sleepless nights in trying to prove FLT.

In spite of that a question still remains: whether Fermat really knew the proof of FLT. Today nothing can be said with certainty about that, but it is certain that so long as human civilization lasts, Wiles will be remembered in the same manner as the FLT which he first proved.

Suggested Reading

[1] Stuart Hollingdale, *Makers of Mathematics*, Penguin Books, 1989.

[2] S Barnard and J M Child, *Higher Algebra*, Macmillan Company, 2000.

[3] F Cajori, *A History of Mathematics*, Macmillan Company, 1909.

[4] R S Calinger (Ed.), *Classics of Mathematics*, p.337, 1994.

[5] D Halliday and R Resnick, *Physics*, Vol.II, Wiley Eastern Ltd. & New Age International, 1994.

[6] V M Tikhomirov, *Stories About Maxima and Minima*, Universities Press, 1998.

[7] Kenneth A Ribet and Brian Hayes, Fermat's Last Theorem and Modern Arithmetic, *American Scientist*, March–April, 1994.

[8] Simon Singh and Kenneth A Ribet, Fermat's Last Stand; *Scientific American*, November, 1997.

[9] S Balachandra Rao, *Indian Mathematics and Astronomy, Some Landmarks*, Jnana Deep Publications, Bangalore, 1994.

[10] Sailesh Dasgupta, After Thought of Wiles' Success; *Indian Journal of Mathematics Teaching*, Vol.XXIII, No.2.

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