Classroom

In this section of *Resonance,* we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

Fluid Dynamics of a Liquid in a U-shaped Tank

Seok-In Hong
Department of Science Education
Gyeongin National University of Education
Anyang, 430-739, Korea.
Email: sihongtao@hanmail.net

Using the unsteady Bernoulli equation, we find solutions for the oscillatory motion of an incompressible and non-viscous liquid in a U-shaped tank. Alternative expressions for height-vs-time relation and the period are obtained through a transformation formula of the elliptic integral of the second kind. From the viewpoint of the mechanical energy conservation, the motion of the free surface can be interpreted as the motion of one particle in a restoring effective potential. This article provides an instructive example of the application of unsteady Bernoulli equation at an undergraduate level.

The U-shaped tank (U-tank) is widely used in a variety of practical applications: A manometer consists of a U-tube containing liquid such as mercury or water [1]. A U-shaped viscometer is used to determine viscosity of both Newtonian [2] and non-Newtonian fluids, including blood [3,4]. The roll motion of a ship can be stabilized by water inside a pair of tanks (known as the U-tank system) which are connected by a water duct at their base [5]. This system is also employed to damp the

Keywords
Fluid dynamics, U-shaped tank, Bernoulli equation.

sway motion of a tall building by wind or earthquake.

It is not possible to solve the full Navier–Stokes equation analytically for the U-tank problem. As mentioned in [1], different approximations of the Navier–Stokes equation are allowed region by region inside the U-tank, which must be matched at the boundaries of adjacent regions. In this article the unsteady Bernoulli equation (energy method) is adopted as an approximation from which we obtain the governing equation for the liquid level (height) in the U-tank with arbitrary cross-sectional areas. In addition, we discuss various aspects of the problem (e.g., the alternative forms of the height-vs-time relation and the period, and the restoring effective potential). This work allows undergraduate students to exploit the energy method and to gain some insights into, and better understanding of the pedagogically useful topic [1,6,7] in fluid mechanics.

Let us consider a U-tank (*Figure* 1) with two vertical columns connected by a horizontal duct which contains an incompressible, and frictionless liquid. Friction losses will not be negligible, especially at bends involving area changes. The cross-sectional areas of the left column, the right column, and the horizontal duct are A_1, A_2, and A_3, respectively. For convenience, the downward vertical direction is chosen as the positive z-axis with the origin at the center-line of the horizontal duct. $H_1(t)$ and $H_2(t)$ are the heights of the left and right free surfaces from that center-line at time t respectively, and D is the length of the horizontal duct. Assume that $H_1(0) \equiv H_{10} > H_2(0) \equiv H_{20} > 0$ without loss of generality.

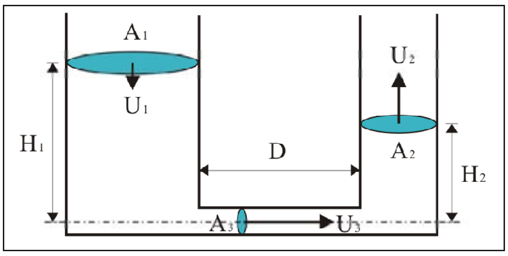

Figure 1. Liquid motion in a U-tank.

By the conservation of energy, an oscillatory periodic motion of the liquid between the two columns is expected, which is obviously an unsteady process. The unsteady Bernoulli equation [8] in this case is given by

$$\int \left(\frac{U^2}{2} - gz \right) \mathrm{d}V = \text{constant}, \qquad (1)$$

where the integration is performed over the volume V of the liquid and only kinetic and gravitational potential energy terms are retained, and the internal energy term is neglected. In particular, we neglect frictional losses in the U-tank. Suppose the liquid velocity U_i $(i = 1,\ 2,\ 3)$ in each region (the left and right vertical columns and the horizontal duct) is uniform along vertical and horizontal directions, and depends only on time:

$$U_i(t) = -\frac{\mathrm{d}H_i(t)}{\mathrm{d}t}, \quad (i = 1,\ 2). \qquad (2)$$

They are related to each other by the conservation of mass

$$A_1|U_1| = A_2|U_2| = A_3|U_3|. \qquad (3)$$

We consider only the case where $H_1(t),\ H_2(t) \geq 0$ for all time t. That is, assume that the liquid in one column is not completely drained into the other column. Hence the horizontal duct is always completely filled with liquid without any void.

Substituting (2) into (1) and using (3) with initial conditions $H_1(0) = H_{10}$, $H_2(0) = H_{20}$, and $U_1(0) = U_2(0) = 0$, we have (see Appendix A):

$$A_1 \left[H_1 \left(\frac{\mathrm{d}H_1}{\mathrm{d}t} \right)^2 + gH_1^2 \right] + \frac{DA_1^2}{2A_3} \left(\frac{\mathrm{d}H_1}{\mathrm{d}t} \right)^2$$
$$+A_2 \left[H_2 \left(\frac{\mathrm{d}H_2}{\mathrm{d}t} \right)^2 + gH_2^2 \right] + \frac{DA_2^2}{2A_3} \left(\frac{\mathrm{d}H_2}{\mathrm{d}t} \right)^2$$
$$= g \left(A_1 H_{10}^2 + A_2 H_{20}^2 \right). \qquad (4)$$

Eliminating H_2 in this equation by using the conservation of mass (3) leads to the governing equations

$$\begin{aligned} \left(\frac{\mathrm{d}H_1}{\mathrm{d}t}\right)^2 &= \frac{g\,(1+\alpha)\,(H_{10}-H_1)(H_1-b)}{(1-\alpha^2)H_1+\alpha(\alpha H_{10}+H_{20})+\beta D} &(5)\\ &= \left(\frac{g}{\alpha-1}\right)\frac{(H_{10}-H_1)(H_1-b)}{\lambda-H_1}, &(6)\\ H_2 &= H_{20}+\alpha\,(H_{10}-H_1)\ , &(7) \end{aligned}$$

where $\alpha \equiv A_1/A_2(>0)$, $\beta \equiv A_1/A_3(>0)$, and

$$\begin{aligned} b &\equiv \frac{(\alpha-1)H_{10}+2H_{20}}{\alpha+1} \quad (< H_{10}),\\ \lambda &\equiv \frac{\alpha^2 H_{10}+\alpha H_{20}+\beta D}{\alpha^2-1}. \end{aligned} \tag{8}$$

Equation (6) is valid for all $\alpha > 0$; the case $\alpha = 1$ is regarded as a limit with $\alpha \to 1$. By differentiating (4) with respect to time, we have another form of the governing equation:

$$\begin{aligned} &\left(H_1+\frac{DA_1}{2A_3}\right)\frac{\mathrm{d}^2H_1}{\mathrm{d}t^2}+\frac{1}{2}\left(\frac{\mathrm{d}H_1}{\mathrm{d}t}\right)^2+gH_1\\ &=\left(H_2+\frac{DA_2}{2A_3}\right)\frac{\mathrm{d}^2H_2}{\mathrm{d}t^2}+\frac{1}{2}\left(\frac{\mathrm{d}H_2}{\mathrm{d}t}\right)^2+gH_2. \end{aligned} \tag{9}$$

Equation (9) may be obtained using the standard form of the unsteady Bernoulli equation along a streamline[1].

[1] See J H Arakeri, Bernoulli's Equation, *Resonance*, Vol.5, No.8, pp.54–71, 2000.

Note that (4) and (9) are symmetric under the exchange of the indices 1 (the left column) and 2 (the right column).

As a special case, consider the case $\alpha = 1$ (i.e., $A_1 = A_2$). From (5) or (9), we have

$$\ell_{\mathrm{eff}}\frac{\mathrm{d}^2H_1}{\mathrm{d}t^2}+g\left(H_1-\frac{H_{10}+H_{20}}{2}\right)=0, \tag{10}$$

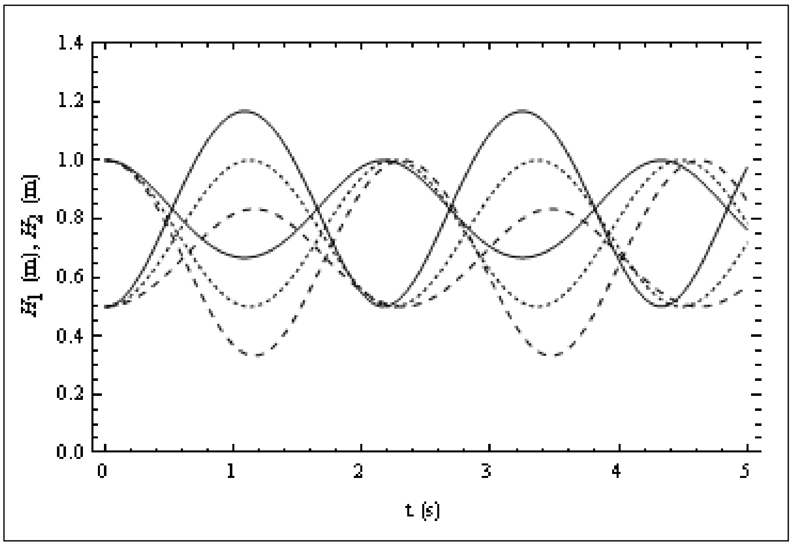

Figure 2. Graphs of H_1 and H_2 as functions of time for α = 2 (full line), α = 1 (dotted line), and α = 1/2 (dashed line). Other parameters are: β = 10, H_{10}= 1 m, H_{20}= 1/2 m, and D = 1/10 m in Figures 2, 3, and 4. For α = 2, we have b = 2/3 m, λ = 2m, H_{2max}= 7/6 m, k = 1/4, and T $\approx$ 2.17s. For α = 1, b = 1/2m, H_{2max} = 1m, k = 0, and T $\approx$ 2.24s. For α = 1/2, b = 1/3m, λ = –2m, H_{2max}= 5/6 m, k = –2/7, and T $\approx$ 2.32 s.

which gives the simple harmonic solution (see *Figure* 2):

$$H_1(t) = \frac{H_{10} + H_{20}}{2} + \left(\frac{H_{10} - H_{20}}{2}\right) \cos \omega t, \qquad (11a)$$

$$H_2(t) = \frac{H_{10} + H_{20}}{2} - \left(\frac{H_{10} - H_{20}}{2}\right) \cos \omega t\,, \qquad (11b)$$

with the effective length $\ell_{\text{eff}} \equiv (H_{10} + H_{20} + \beta D)/2$ and $\omega \equiv \sqrt{g/\ell_{\text{eff}}}$. Thus the case $\alpha = 1$ (the motion of H_1 with equilibrium position $(H_{10} + H_{20})/2$) is equivalent to a small-angle simple pendulum whose length is ℓ_{eff}. Note that (10) is consistent with the well-known result [6,7] for the simplest case $\alpha = \beta = 1$ (for example, a U-tube manometer of constant diameter).

Now let us find the exact solution for arbitrary α (> 0). From (6), $U_1 = 0$ at $H_1 = H_{10}$ and $H_1 = b$, and hence $b \leq H_1 \leq H_{10}$. If $\alpha > 1$, then $b > 0$. However, if $0 < \alpha < 1$, H_{10} and H_{20} must satisfy $H_{20} \geq (1-\alpha)H_{10}/2$ for $b \geq 0$. Otherwise, the left tank will be completely drained and the horizontal duct will have an empty region, for which the governing equation must be modified. This case will not be treated here. $\lambda > H_{10} > 0$

The analysis in this article gives undamped oscillations as viscous losses have been neglected.

for $\alpha > 1$, while $\lambda < 0$ for $0 < \alpha < 1$. When $b \geq 0$, b is the lowest point of the left free surface. The corresponding maximum height of H_2 is given by $H_{2\max} = H_{20} + \alpha(H_{10} - b)$.

By a change of variable

$$h \equiv \sqrt{\frac{H_1 - b}{H_{10} - b}}, \qquad (0 \leq h \leq 1), \tag{12}$$

(6) is rewritten as

$$\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2 = \frac{g}{4p(\alpha)}\left(\frac{1-h^2}{1-k(\alpha)h^2}\right), \tag{13}$$

where

$$p(\alpha) \equiv (\alpha - 1)(\lambda - b), \quad k(\alpha) \equiv \frac{H_{10} - b}{\lambda - b}. \tag{14}$$

Note that $p(\alpha = 1) \equiv \lim_{\alpha\to 1} p(\alpha) = (H_{10}+H_{20}+\beta D)/2$ and $k(\alpha = 1) \equiv \lim_{\alpha\to 1} k(\alpha) = 0$, while $p > 0$ for all $\alpha > 0$, $0 < k < 1$ for $\alpha > 1$, and $k < 0$ for $0 < \alpha < 1$. Equation (13) may be integrated to

$$\int \sqrt{\frac{1-kh^2}{1-h^2}}\mathrm{dh} = \mp\sqrt{\frac{g}{4p}}\int \mathrm{d}t, \tag{15}$$

where minus and plus signs in the right-hand side correspond to the downward (descending) and upward (ascending) motions of H_1, respectively.

Let us introduce the incomplete elliptic integral of the second kind $E(\phi|\mu)$ which is defined by

$$E(\phi|\mu) \equiv \int_0^{\phi} \sqrt{1-\mu\sin^2\theta}\,\mathrm{d}\theta \;=\; \int_0^{\sin\phi} \sqrt{\frac{1-\mu x^2}{1-x^2}}\,\mathrm{d}x, \quad (\mu \leq 1). \tag{16}$$

$E(\pi/2|\mu) \equiv E(\mu)$ is the complete elliptic integral of the second kind.

The period T of the liquid motion can be calculated by integrating from $t = 0$ $(h = 1)$ to $t = T/2$ $(h = 0)$ in (15) with the minus sign:

$$T = 4\sqrt{\frac{p}{g}}\, E\,(k)\,. \tag{17}$$

Combining the first descending solution $(0 \le t \le T/2)$ and the first ascending solution $(T/2 \le t \le T)$, and considering the periodic property of the motion, we have the solution for all time t:

$$E\left(\arcsin\sqrt{\frac{H_1 - b}{H_{10} - b}}\,\middle|\, k\right) = E(k)\tau(t), \tag{18}$$

with periodic function $\tau(t)$ defined by $\tau(t) = |1 - (2t/T)|$ for $0 \le t \le T$ and $\tau(t) = \tau(t - T)$ for $t \ge T$. See Appendix B.

In the case of $0 < \alpha < 1$, $k < 0$ for $b \ge 0$. Though (17) and (18) are valid for negative k (i.e., $k = -|k| < 0$), they can be transformed to the alternative expressions that contain the elliptic integral of the second kind with a positive parameter k' defined by $k' \equiv |k|/(1 + |k|) = (H_{10} - b)/(H_{10} - \lambda)$. With the aid of the formula

$$E(\phi|k) = E(k) - \sqrt{1 + |k|}\; E\,(\pi/2 - \phi|k') \tag{19}$$

and its special case $(\phi = 0)$

$$E(k) = \sqrt{1 + |k|}\; E\,(k')\,, \tag{20}$$

(17) and (18) reduce to

$$T = 4\left[\frac{(1-\alpha)(H_{10} - \lambda)}{g}\right]^{1/2} E\,(k')\,, \tag{21}$$

$$E\left(\arcsin\sqrt{\frac{H_{10} - H_1}{H_{10} - b}}\,\middle|\, k'\right) = E\,(k')\,[1 - \tau(t)]\,. \tag{22}$$

For the derivation of (22), note that (18) can be rewritten as

$$E\left(\frac{\pi}{2}-\arcsin\sqrt{\frac{H_1-b}{H_{10}-b}}\,\middle|k'\right)=\frac{E(k)}{\sqrt{1+|k|}}\left[1-\tau(t)\right], \tag{23}$$

by using (19). This leads to (22) immediately by simple trigonometry and (20).

Graphs of $H_1(t)$ and $H_2(t)$ as functions of time are shown in *Figure* 2 for $\alpha = 2,\ 1,\ 1/2$. Other parameters are given by $\beta = 10$, $H_{10} = 1$ m, $H_{20} = 1/2$ m, and $D = 1/10$ m in *Figures* 2, 3, and 4.

The liquid velocity U_1 in the left vertical column can be obtained from (2) and (6) with plus and minus signs of U_1 being downward and upward motions, respectively. Note that $U_2 = -\alpha U_1$ and $U_3 = \beta U_1$. The speed of the left free surface has the maximum value

$$|U_1|_{\max}=\left[\pm\left(\frac{g}{\alpha-1}\right)\frac{(H_{10}-\lambda\pm\sigma)(\lambda-b\mp\sigma)}{\sigma}\right]^{1/2}, \tag{24}$$

where $\sigma \equiv [\lambda(\lambda-b-H_{10})+bH_{10}]^{1/2}$, at $H_1 = \lambda \mp \sigma$, where the upper signs and the lower signs correspond to the cases $\alpha \geq 1$ and $0 < \alpha \leq 1$, respectively. *Figure* 3 shows the graphs of (a) U_1 as functions of t and (b) $|U_1|$ as functions of H_1, for $\alpha = 2, 1, 1/2$. The velocity $U_1(t)$

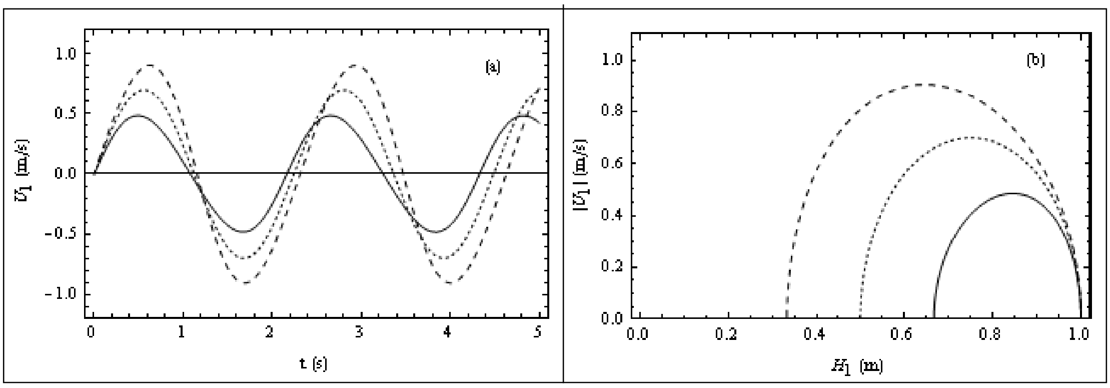

Figure 3. (a) Graphs of U_1 as functions of t and (b) graphs of $|U_1|$ as functions of H_1, for $\alpha = 2$ (full line), $\alpha = 1$ (dotted line), and $\alpha = 1/2$ (dashed line), where the speed $|U_1|$ of the left free surface has the maximum values 0.48ms^{-1}, 0.70 ms^{1}, ~0.91ms^{-1} at the heights $H_1 = 2 - 2\sqrt{3}$ ($\approx$ 0.85) m, 3/4m, $\sqrt{7} - 2$ ($\approx$ 0.65) m, respectively. The motion of H_1 can be interpreted as one-particle motion with arbitrary mass and zero energy in the potential well $W(H_1) = -U_1^2(H_1)/2$ (see equation (27)).

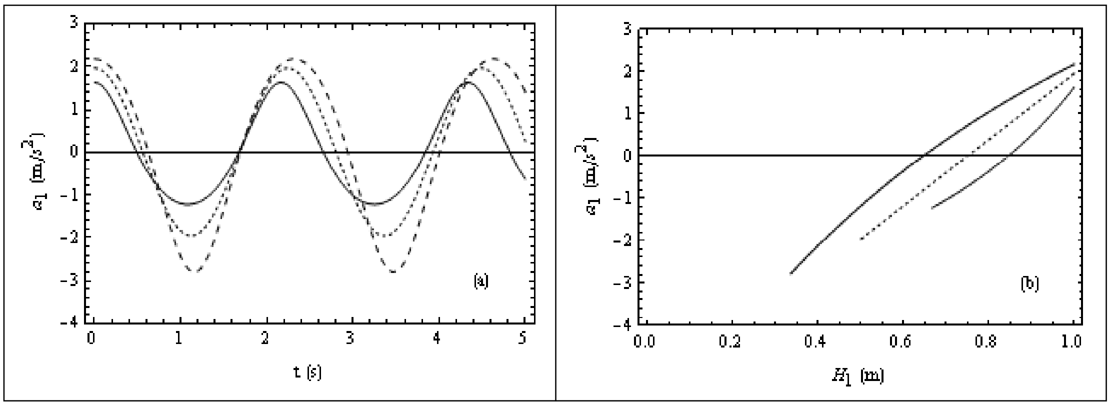

Figure 4. Graphs of a_1 as functions of (a) t and (b) H_1. For the three cases α = 2 (full line), α =1 (dotted line), and α =1/2 (dashed line), (a_1 (H_{10}),a_1 (b)) has the values (1.63ms^{-2},–1.23ms^{-2}), (1.96ms^{-2},–1.96ms^{-2}), and (2.18ms^{-2},–2.80ms^{-2}), respectively.

(*Figure* 3a) and the acceleration $a_1(t)$ (*Figure* 4a) as functions of t are obtained by substituting $H_1(t)$ found numerically from (18) into $U_1(H_1)$ and $a_1(H_1)$ as functions of H_1, respectively.

To examine the initial behavior of the liquid motion, put $H_1(t) = H_{10} - \delta H_1$ with $0 \le \delta H_1 \ll H_{10}$. Then from (6) we obtain $U_1 \approx \sqrt{2\gamma g(\delta H_1)} \approx \gamma g t$, where $\gamma \equiv (H_{10} - H_{20})/(H_{10} + \alpha H_{20} + \beta D)$ and the second approximation comes from $\delta H_1 \approx a_1(H_{10})t^2/2$ with the initial acceleration $a_1(H_{10})$ given by (26).

The acceleration of the left free surface can be calculated by differentiating (6) with respect to time:

$$a_1(H_1) = -\frac{g}{2(\alpha-1)}\left[\frac{H_1^2 - 2\lambda H_1 + \lambda(H_{10}+b) - bH_{10}}{(H_1-\lambda)^2}\right]. \tag{25}$$

As special cases,

$$a_1(H_{10}) = \frac{g}{2(\alpha-1)}\left(\frac{H_{10}-b}{\lambda - H_{10}}\right),$$
$$a_1(b) = -\frac{g}{2(\alpha-1)}\left(\frac{H_{10}-b}{\lambda - b}\right). \tag{26}$$

As expected, $|a_1(H_{10})|$ is greater than, equal to, less than $|a_1(b)|$ for $\alpha > 1$, $\alpha = 1$, $0 < \alpha < 1$, respectively. *Figure* 4 displays the graphs of a_1 as functions of (a) time t and (b) height H_1, for $\alpha = 2, 1, 1/2$. Note that from (11a) or (25), a_1 is linear in H_1 for $\alpha = 1$: $a_1 = \omega^2 [H_1 - (H_{10} + H_{20})/2]$.

As a final remark, the liquid motion (specifically, the motion of the left free surface) can be interpreted in terms of *one-particle* motion. Let H_1 be the position of a fictitious particle with arbitrary mass. Equation (6) represents the mechanical energy conservation (per unit mass) of that particle with zero energy in the potential well (see *Figure* 3b):

$$W(H_1) = -\frac{1}{2}U_1^2 = \frac{g}{2(\alpha - 1)} \left[\frac{(H_1 - H_{10})(H_1 - b)}{\lambda - H_1} \right]. \qquad (27)$$

Note that the potential seen by the fictitious particle depends on the initial condition of the U-tank problem and is zero at the stationary points of the particle. When $\alpha = 1$, (27) reduces to a quadratic potential

$$W(H_1) = \frac{1}{2}\omega^2 (H_1 - H_{10})(H_1 - H_{20}), \qquad (28)$$

which in turn leads to the equation of motion (10).

Appendix A: Derivation of Equation (4)

The integration [$\equiv I(t)$] in (1) can be divided into three parts according to regions (the left and right vertical columns and the horizontal duct) of the liquid in the U-tank:

$$I(t) = I_1(t) + I_2(t) + I_3(t) = \left(\int_{\text{Left}} + \int_{\text{Right}} + \int_{\text{Horizontal}} \right) \left(\frac{U^2}{2} - gz \right) \mathrm{d}V. \qquad (\text{A1})$$

Note that the potential energy of the liquid in the horizontal duct whose centerline is located at $z = 0$ is zero. Assuming uniform velocity profile in each region, we have

$$I_1 = \int_{-H_1}^{0} \left(\frac{U_1^2}{2} - gz \right) A_1 \mathrm{d}z = \frac{A_1}{2} \left(H_1 U_1^2 + g H_1^2 \right), \qquad (\text{A2})$$

$$I_2 = \frac{A_2}{2}\left(H_2U_2^2 + gH_2^2\right), \tag{A3}$$

$$I_3 = \int_0^D \frac{U_3^2}{2}A_3\mathrm{d}x = \frac{U_3^2 A_3 D}{2} = \frac{D}{4A_3}\left[(U_1A_1)^2 + (U_2A_2)^2\right], \tag{A4}$$

where (3) is used for the last equality in (A4). Hence we obtain (4) by substituting (2) into these results and taking $I(0)$ as the constant in (1).

Appendix B. Derivation of Equation (18)

For the first descending solution, we use (15) with the minus sign, (16), and (17):

$$\int_1^h \sqrt{\frac{1-kh^2}{1-h^2}}\mathrm{d}h = \left(\int_0^h - \int_0^1\right)\sqrt{\frac{1-kh^2}{1-h^2}}\mathrm{d}h = -\frac{2E(k)}{T}\int_0^t \mathrm{d}t, \tag{B1}$$

which gives

$$E\left(\arcsin\sqrt{\frac{H_1-b}{H_{10}-b}}\,\middle|\,k\right) = E(k)\left(1-\frac{2t}{T}\right), \quad (0 \le t \le T/2). \tag{B2}$$

Similarly, the first ascending solution can be found by using (15) with the plus sign:

$$\int_0^h \sqrt{\frac{1-kh^2}{1-h^2}}\mathrm{d}h = +\frac{2E(k)}{T}\int_{T/2}^t \mathrm{d}t, \tag{B3}$$

which leads to

$$E\left(\arcsin\sqrt{\frac{H_1-b}{H_{10}-b}}\,\middle|\,k\right) = -E(k)\left(1-\frac{2t}{T}\right), (T/2 \le t \le T). \tag{B4}$$

These two solutions can be combined to give (18).

Suggested Reading

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