A Pizza Saga

Five Delicacies of Modern Geometry

Shailesh A Shirali

In this article we introduce the reader to some pretty results concerning a circle and regions within it defined by various line segments. We have framed the results in terms of a pizza, so we could well call this an introduction to *Pizza Geometry*! The novel aspect is the variety of approaches used to prove these results.

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1. First Pizza Theorem

Shown in *Figure* 1 is a circular region representing a pizza. Through an arbitrary point P within it, four chords AE, BF, CF and DG are drawn such that the angle between each pair of adjacent chords is 45°. This creates eight wedge-shaped regions PAB, PBC, PCD, ... (eight wedges of the pizza), of which, typically, no two regions will be congruent to each other (unless P lies at the center of the circle). Let the regions be coloured alternately red and green, as shown in *Figure* 1.

The first pizza theorem states: *The total area of the green regions equals the total area of the red regions, and this is so regardless of where* P *is located within the circle.*

We shall give a delightful calculus-based proof of this result, drawing on results from circle geometry; the proof is from [2]. The basic lemma used is the following: *Let* P *be a point within a circle* (O, R), *and let* AC *and* BD *be a pair of perpendicular chords of the circle, intersecting at* P. *Then* $\mathrm{PA}^2 + \mathrm{PB}^2 + \mathrm{PC}^2 + \mathrm{PD}^2 = 4R^2$ (*Figure* 2).

To see why, note that $\mathrm{PA}^2 + \mathrm{PB}^2 + \mathrm{PC}^2 + \mathrm{PD}^2 = \frac{1}{2}(\mathrm{AB}^2 + \mathrm{BC}^2 + \mathrm{CD}^2 + \mathrm{DA}^2)$. Next, from $\mathrm{AC} \perp \mathrm{BD}$ we get $\angle\mathrm{AOB} + \angle\mathrm{COD} = 180°$; i.e., $\angle\mathrm{AOB}$ and $\angle\mathrm{COD}$ are

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Pizza theorem, Euler–Poincaré formula, Ptolemy's theorem, vector formula for area.

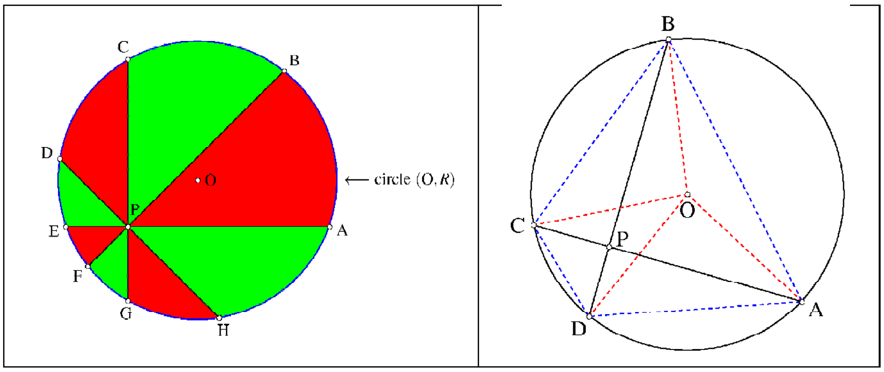

Figure 1. (left) Figure for first pizza theorem.

Figure 2. (right) Lemma concerning two perpendicular chords.

supplementary angles. Since $\mathrm{AB} = 2R \sin \frac{1}{2}\angle \mathrm{AOB}$, and $\mathrm{CD} = 2R \sin \frac{1}{2}\angle \mathrm{COD} = 2R \cos \frac{1}{2}\angle \mathrm{AOB}$, we get $\mathrm{AB}^2 + \mathrm{CD}^2 = 4\mathrm{R}^2$. In the same way we get $\mathrm{BC}^2 + \mathrm{AD}^2 = 4R^2$, and the stated result follows.

Referring to *Figure* 1, let P be the pole of a polar coordinate system, with ray PA as the axis; let the equation of the circle be $r = f(\theta)$. Then the above lemma may be expressed in terms of f as follows:

$$f(\theta)^2 + f\left(\theta + \frac{\pi}{4}\right)^2 + f\left(\theta + \frac{\pi}{2}\right)^2 + f\left(\theta + \frac{3\pi}{4}\right)^2 = 4R^2, \text{ for all } \theta. \quad (1)$$

With the vertices labeled as shown in *Figure* 1 (i.e., counter-clockwise), the total area of the red region is

$$\int_0^{\pi/4} \frac{1}{2} f(\theta)^2 \,\mathrm{d}\theta + \int_{\pi/2}^{3\pi/4} \frac{1}{2} f(\theta)^2 \,\mathrm{d}\theta + \int_{\pi}^{5\pi/4} \frac{1}{2} f(\theta)^2 \,\mathrm{d}\theta + \int_{3\pi/2}^{7\pi/4} \frac{1}{2} f(\theta)^2 \,\mathrm{d}\theta$$

$$= \int_0^{\pi/4} \frac{1}{2}\left[f(\theta)^2 + f\left(\theta + \frac{\pi}{4}\right)^2 + f\left(\theta + \frac{\pi}{2}\right)^2 + f\left(\theta + \frac{3\pi}{4}\right)^2\right] \mathrm{d}\theta.$$

We give a calculus-based proof of the theorem.

Using relation (1) this reduces to (as claimed).

$$\int_0^{\pi/4} 2R^2\,\mathrm{d}\theta = \frac{1}{2}\pi R^2 = \text{half the area of the circle.}$$

2. Second Pizza Theorem

For the second theorem, we start once again with a circular region representing a pizza, and select an arbitrary point P within it. This time we mark eight points A, B, C, D, E, F, G, H on the periphery of the circle at regular intervals (so that angles AOB, BOC, COE, … , HOA are all equal to 45°). We now join P to each of these points, thereby creating eight wedge-shaped regions PAB, PBC, PCD, … (eight wedges of the pizza), of which, typically, no two will be congruent to each other (unless P lies at the center of the circle). As earlier, we colour these regions alternately red and green, as shown in *Figure* 3.

The second pizza theorem reads the same as the first one.

The second pizza theorem, remarkably, reads the same as the first one: *The total area of the green regions equals the total area of the red regions, and this is so regardless of where* P *is located within the circle.*

It is clearly sufficient to show that this equality of areas is true for the regular polygon ABCDEFGH rather than the circle (see *Figure* 4), and this is what we shall

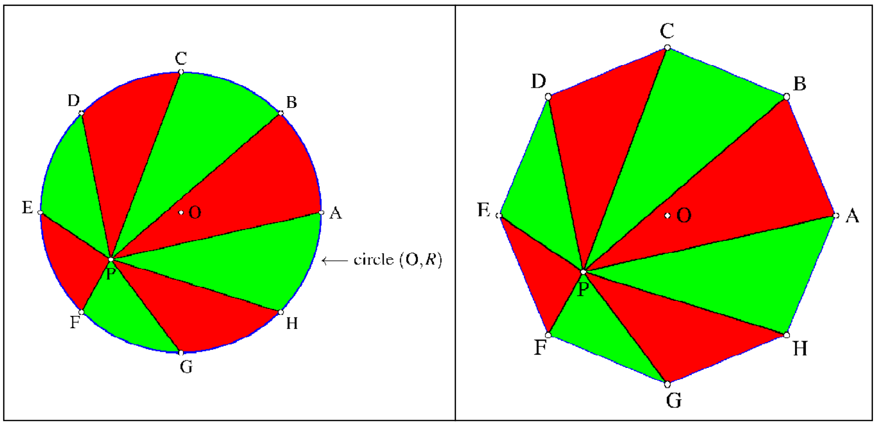

Figure 3. (left) Figure for second pizza theorem.
Figure 4. (right).

do, using vector methods. With O as the origin, let the position vectors of the various points be indicated using boldface symbols, thus: **A**, **B**, **C**, **D**, **E**, **F**, **G**, **H**, **P**; these vectors have equal magnitude, and the angles between **A** and **B**, between **B** and **C**, between **C** and **D**, etc., are equal to one another. We use the following well-known formula: *Given three points* K, L, M*, with position vectors* **K***,* **L***,* **M***, respectively, the area of triangle* KLM *is the magnitude of the vector* $\frac{1}{2}(\mathbf{L}-\mathbf{K})\times(\mathbf{M}-\mathbf{K})$. More symmetrically expressed, the area is the magnitude of $\frac{1}{2}(\mathbf{K}\times\mathbf{L}+\mathbf{L}\times\mathbf{M}+\mathbf{M}\times\mathbf{K})$.

So the area of the red region is half the magnitude of the following vector:

$$\begin{aligned}(&\mathbf{P}\times\mathbf{A}+\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{P})\\ &+(\mathbf{P}\times\mathbf{C}+\mathbf{C}\times\mathbf{D}+\mathbf{D}\times\mathbf{P})\\ &+(\mathbf{P}\times\mathbf{E}+\mathbf{E}\times\mathbf{F}+\mathbf{F}\times\mathbf{P})\\ &+(\mathbf{P}\times\mathbf{G}+\mathbf{G}\times\mathbf{H}+\mathbf{H}\times\mathbf{P}).\end{aligned} \tag{2}$$

The above expression simplifies to

$$\begin{aligned}&\mathbf{P}\times(\mathbf{A}-\mathbf{B}+\mathbf{C}-\mathbf{D}+\mathbf{E}-\mathbf{F}+\mathbf{G}-\mathbf{H})\\ &+(\mathbf{A}\times\mathbf{B}+\mathbf{C}\times\mathbf{D}+\mathbf{E}\times\mathbf{F}+\mathbf{G}\times\mathbf{H}).\end{aligned} \tag{3}$$

This further simplifies to

$$\mathbf{A}\times\mathbf{B}+\mathbf{C}\times\mathbf{D}+\mathbf{E}\times\mathbf{F}+\mathbf{G}\times\mathbf{H}, \tag{4}$$

because the vector sums $\mathbf{A}+\mathbf{E}$, $\mathbf{B}+\mathbf{F}$, $\mathbf{C}+\mathbf{G}$, $\mathbf{D}+\mathbf{H}$ all vanish (by symmetry), so that $\mathbf{A}-\mathbf{B}+\mathbf{C}-\mathbf{D}+\mathbf{E}-\mathbf{F}+\mathbf{G}-\mathbf{H}$ vanishes as well. *Hence the area of the red region is independent of* P.

In the same way we see that the area of the green region is half the magnitude of the vector

$$\mathbf{B}\times\mathbf{C}+\mathbf{D}\times\mathbf{E}+\mathbf{F}\times\mathbf{G}+\mathbf{H}\times\mathbf{A}, \tag{5}$$

The area of the red region is independent of P.

and it is clear that (4) and (5) represent the same vector (since $\mathbf{A}\times\mathbf{B}=\mathbf{B}\times\mathbf{C}$, etc). So the total areas of the red and green regions are equal to each other, as claimed.

Remark. The two theorems proved above have featured the number 8, but this number can be substituted by any multiple of 8, and the corresponding result will continue to be true.

3. Third Pizza Theorem

For the third theorem, we mark n points on the periphery of the pizza, and run straight cuts through it along the lines joining all possible pairs of points; *Figure* 5 shows the cases $n = 2, 3, 4, 5$. Our intention is to divide the pizza into as many pieces as possible (the pieces are not required to be of equal size, so sympathies need to be offered to those who get the small pieces), so we must position the n points in such a way that no three of the cuts concur. *What is the largest number of pieces we can get?* Let $f(n)$ denote this number. The values of $f(n)$ for n = 1, 2, 3, ... may be found by experimentation (see *Figure* 5), and we get the following table of values:

n	1	2	3	4	5	$\cdots$
$f(n)$	1	2	4	8	16	$\cdots$

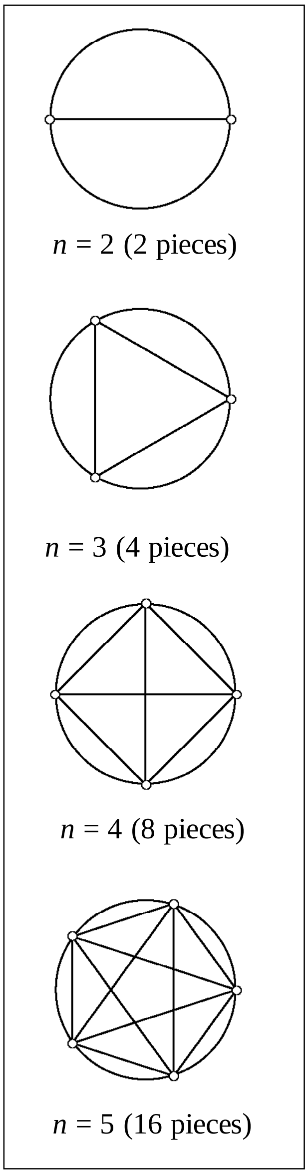

Figure 5. Third pizza theorem.

From this data we readily guess that $f(n) = 2^{n-1}$. *But this guess turns out to be wrong.* In fact, $f(6) \neq 32$.

Instead we have the following remarkably compact formula:

$$f(n) = \binom{n}{4} + \binom{n}{2} + \binom{n}{0}. \qquad (6)$$

For example, $f(4) = \binom{4}{4} + \binom{4}{2} + \binom{4}{0} = 8$, and $f(5) = \binom{5}{4} + \binom{5}{2} + \binom{5}{0} = 16$. These agree with the numbers given above.

Using the identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, we may write (6) in the form

$$f(n) = \binom{n-1}{4} + \binom{n-1}{3} + \binom{n-1}{2} + \binom{n-1}{1} + \binom{n-1}{0}, \qquad (7)$$

The obvious guess for $f(n)$ turns out to be wrong.

To prove the third theorem, we use the Euler–Poincaré formula which is true for any connected planar graph: $v - e + f = 2$.

from which we can see why $f(n) = 2^{n-1}$ for $n - 1 \leq 4$, i.e., for $n \leq 5$. (Recall that for each n the sum $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n}$ equals 2^n.)

To prove formula (6) we shall use the well-known formula $v - e + f = 2$ which holds for any connected planar graph, with v, e, and f representing the numbers of vertices, edges, and faces of the graph respectively[1]. For example, for the case $n = 4$ in *Figure* 5, we have $v = 5$, $e = 12$, $f = 9$, and $5 - 12 + 9 = 2$, as we should expect. (Note that $v = 5$, and not 4, because the intersecting chords have created an additional vertex within the circle. Also, $f = 9$, and not 8, because there is the 'outside' region too to be counted.)

To apply this to our setting, we need the values of v and e. We first compute v. To start with, there are n vertices on the circle. Each vertex within the circle is the intersection of the diagonals of a quadrilateral formed by four vertices on the circle. The number of such quadrilaterals is precisely the number of ways of choosing 4 vertices out of the set of n vertices, hence it is equal to $\binom{n}{4}$. It follows that

$$v = \binom{n}{4} + n. \tag{8}$$

Now we compute e. Visit each vertex of the graph and count the edges emanating from it; then the total count we get in the end will be $2e$, since each edge gets counted twice (once from each end). Since the number of edges emanating from each vertex on the circle is $n - 1$, while the number of edges emanating from each interior vertex is 4, we get the following equality:

$$2e = n(n-1) + 4\binom{n}{4}.$$

It follows that

$$e = \binom{n}{2} + 2\binom{n}{4}. \tag{9}$$

[1] The formula was first proved by Leonhard Euler in the middle of the eighteenth century; but it had been proved earlier by René Descartes, in the seventeenth century, though in a more geometric form (it was with reference to the solid and face angles of a convex polyhedron). See reference 4.} For example, for the case n = 4 in *Figure* 5, we have v = 5, e = 12, f = 9, and 5 - 12 + 9 = 2, as we should expect. (Note that v = 5, and not 4, because the intersecting chords have created an additional vertex within the circle. Also, f = 9, and not 8, because there is the "outside" region too to be counted.

We are now in a position to compute f, using the relation $v - e + f = 2$:

$$f = \binom{n}{2} + 2\binom{n}{4} - \binom{n}{4} - n + 2 = \binom{n}{4} + \binom{n}{2} + 2.$$

The number of regions within the circle is $f - 1$, and hence is equal to

$$\binom{n}{4} + \binom{n}{2} + 1.$$

This may be written in the more pleasing form $\binom{n}{4} + \binom{n}{2} + \binom{n}{0}$, as noted earlier.

For those who got the smaller pieces of the pizza in this unequal division, we trust that this formula offers more than enough compensation.

4. Fourth Pizza Theorem

Yet again we divide the pizza into wedges through a point not at its center, but this time we divide it into 6 pieces, the angle between adjacent cuts being 60° (see *Figure* 6). Let the three cuts be AD, BE, and CF. Then we have the following pleasing relation between the lengths of the six line segments so created:

$$\mathrm{PA} + \mathrm{PC} + \mathrm{PE} = \mathrm{PB} + \mathrm{PD} + \mathrm{PF}. \qquad (10)$$

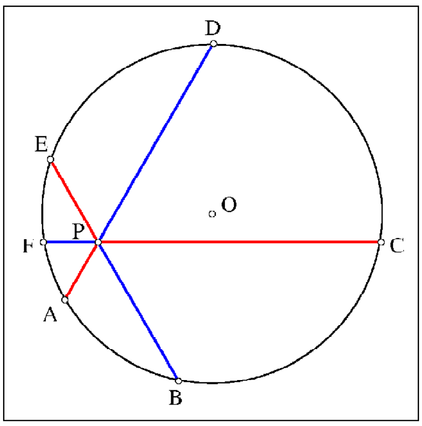

Figure 6. Fourth pizza theorem: PA + PC + PE = PB + PD + PF.

To prove the fourth theorem, we use Ptolemy's theorem of circle geometry.

Different proofs are possible, but we opt to give one using Ptolemy's theorem which states that for a cyclic quadrilateral ABCD we have:

$$\mathrm{AC} \cdot \mathrm{BD} = \mathrm{AB} \cdot \mathrm{CD} + \mathrm{AD} \cdot \mathrm{BC}\ .$$

Let the perpendicular bisectors l_1, l_2, l_3 of chords AD, BE, CF be drawn (see *Figure* 7). The same three lines are obtained no matter where P is located within the circle, and they divide the interior of the circle into 6 congruent sectors. With each point in the circle we associate a 3-tuple of signs, namely, [Sign(PA − PD), Sign(PB − PE), Sign(PC − PF)], where 'Sign(x)' refers to the sign of the real number x. We shall call this the 'signature' of the point P. There are 6 such tuples, one corresponding to each of the 6 sectors.

The argument to prove the theorem has to be phrased slightly differently depending on the sector in which P lies; but the essential logic is the same, so we need to give it for any one sector; there is no loss of generality in doing so. We shall suppose that P lies in the sector with signature $[-,+,+]$.

Let L, M, N denote the midpoints of chords AD, BE, CF, respectively (see *Figure* 8). Then L, M, N lie on the circle

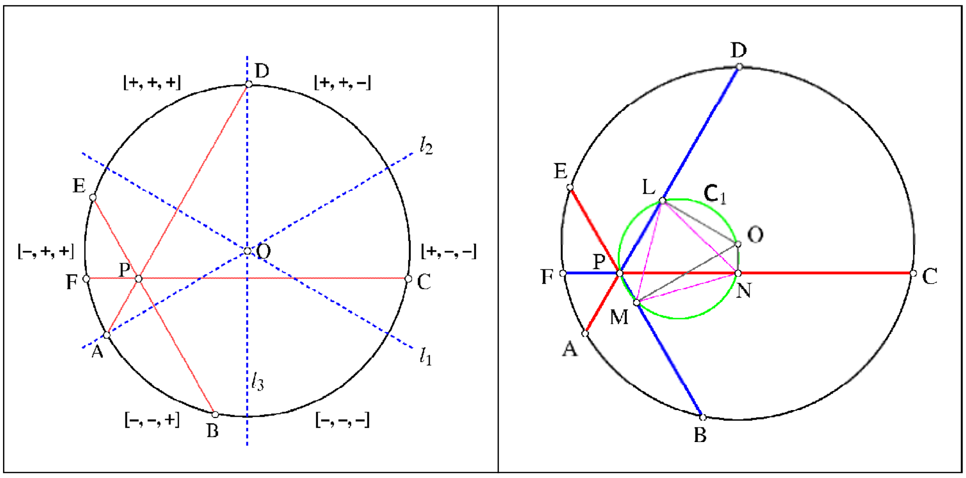

Figure 7. (left) Categorizing six types of regions within the circle.

Figure 8. (right) Proof of fourth pizza theorem, by locating midpoints L, M, N.

C_1 with OP as diameter, and triangle LMN is equilateral. Since P lies in the sector with signature $[-,+,+]$, the picture is as shown in *Figure* 8, with P on the minor arc LM of circle C_1. Ptolemy's theorem applied to quadrilateral PMNL tells us that $\mathrm{PN} \cdot \mathrm{LM} = \mathrm{PL} \cdot \mathrm{MN} +$
(11)

$\mathrm{PM} \cdot \mathrm{NL}$, and since triangle LMN is equilateral, this implies that $\mathrm{PN} = \mathrm{PL} + \mathrm{PM}$. Hence we have:

$$\mathrm{PC} - \mathrm{PF} = (\mathrm{PD} - \mathrm{PA}) + (\mathrm{PB} - \mathrm{PE}),$$

and therefore:

$$\mathrm{PA} + \mathrm{PC} + \mathrm{PE} = \mathrm{PB} + \mathrm{PD} + \mathrm{PF}.$$

5. And a Fifth Pizza Theorem ...

As observed by several writers, there is a *fifth* pizza theorem – and it is the oldest of them all, predating the others by two millennia; maybe predating pizzas as well! This is the result that the volume of a pizza of radius z and thickness a is given by the expression

$$\mathrm{pi}\, zz\ a.$$

Happy eating!

Suggested Reading

[1] http://en.wikipedia.org/wiki/Pizzatheorem

[2] Michael Jeremy, Andrew K Jeremy, and Philip Hirschhorn. The pizza theorem, *Austral. Math. Soc. Gaz.*, Vol.26, pp.120–121, 1999. http://www.maths.unsw.edu.au/~mikeh/webpapers/paper57.pdf.

[3] Rick Mabry and Paul Deiermann, Of Cheese and Crust: A Proof of the Pizza Conjecture and Other Tasty Results, *American Mathematical Monthly*, Vol.116, No.5, pp.423–438, 2009. This reference extends the (first) pizza theorem in a pretty and non-obvious way, and is highly recommended. The article is available at: http://www.lsus.org/sc/math/rmabry/pizza/Pizza_Conjecture.pdf.

[4] http://en.wikipedia.org/wiki/Euler_characteristic

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