Kempe's Linkages and the Universality Theorem

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Inspired by James Watt's approximate straight line generator, kinematicians of the 19th century challenged themselves to design a mechanical device that could convert rotary motion into a perfect straight line and vice versa. Few inventions emerged in 1864 due to Peaucellier and Lipkin and in 1875 due to Hart. Just a year later, in 1876, Alfred B Kempe presented a generalized method for linkages that could exactly trace any algebraic curve of degree n and not just a straight line. This work of Kempe is of classical importance. Yet, many are not aware of it perhaps because the resulting linkages are quite complex. This article discusses Kempe's method that highlights the way he treated the rotations analytically using only parallelograms and contra-parallelograms to get the final rigid-body linkage tracing a given algebraic curve. An elaborate example with geometric construction using only a ruler and compass is presented to help the readers understand the assembly of Kempe's linkages.

1. Introduction

Kempe's work of 1876, now known as the *Universality Theorem*, has a distinctive standing in kinematics. He was the first to uniquely address the precise tracing of any algebraic curve through the geometry of linkwork. Interestingly, his work appeared after researchers had just learnt to draw a straight line exactly through links that are hinged to one another. It is often stated that a Kempe's mechanism can trace one's signature if it is modeled as a continuous curve $f(x, y) = 0$.

Keywords
Kempe's linkages, path generation, Universality Theorem, additor, multiplicator, reversor.

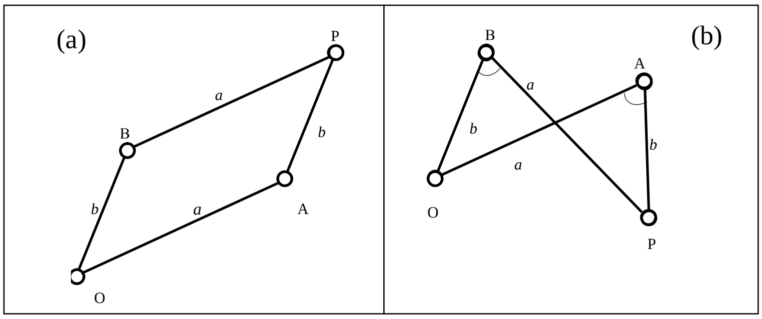

***Figure 1 . (a) Parallelogram* OAPB *and (b) contra-parallelogram* OBAP.**

A simple transformation from Cartesian (i.e., x, y) to polar (i.e, r, θ) coordinates allows $f(x, y)$ to be expressed as a sum of the cosines (e.g., $\cos(\theta)$, $\cos(2\theta)$, etc.). Kempe's ingenuity was in identifying a mechanical linkage, composed of only parallelograms (*Figure* 1a) and contra-parallelograms (*Figure* 1b), for each term in the summation. Each cosine term operates on (i) an angle multiplied by an integer and/or (ii) two angles added together or one subtracted from the other. Both angles can be variable or one of them can be constant. To double the angle made by a link with the horizontal, Kempe proposed to use two aptly connected contra-parallelograms such that another link makes double the angle. He called this two contra-parallelogram linkage a *reversor*. A simple extension allowed multiplying an angle by any integer through the use of similar contra-parallelograms. To add the angles made by two links with the horizontal, Kempe coupled two reversors. A triangular body was used to add or subtract an angle. Finally, all cosine terms in the expansion of $f(x, y)$ were added by translating the corresponding links via a connected chain of parallelograms. Kempe's ability to visualize component linkages hidden beneath the expression of a generic algebraic curve through the use of high school geometry and trigonometry was simply, remarkable!

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Kempe's proposition has incredible theoretical significance but has not been pursued as extensively as much simpler linkages like the four-bar (four links connected with four pin joints) mechanisms have been. Kempe's linkages can be bulky and complex as they usually have many links connected to each other via numerous hinges. Kempe promptly acknowledged the complexity of the linkage "*as a necessary consequence of the perfect generality of demonstration.*" He further stated that "*there is a way of drawing any given case; and the variety of methods of expressing particular functions that have already been discovered renders it in the highest degree probable that in every case a simpler method can be found. There is still, therefore, a wide field open to the mathematical artist to discover the simplest link-works that will describe particular curves.*"

About a hundred years ago, researchers were not blessed with the computational amenities one has today. It was therefore cumbersome to conceive such a linkage on paper through the use of conventional drawing methods. Fabricating a Kempe's linkage and witnessing it in action was far from reality then, something that can be easily accomplished in a computer simulation with a few mouse clicks today. One of the aims of this article is to help the reader understand the construction of Kempe's linkage step-by-step. This can be accomplished by using fundamental ideas from geometry and trigonometry. Description of Kempe's method is incomplete without an example and hence an illustration of a linkage that traces a conic[1] is provided. Individual parts of Kempe's linkages are constructed using the conventional compass–ruler–paper approach, just as it may have been done around 1876. Grübler's criterion from the kinematics literature is also used to count the degrees of freedom in different linkages of a Kempe's mechanism. Given the number of links and hinge joints, Grübler's criterion helps determine the number of minimal inputs a linkage needs for its complete description. These are called the degrees of freedom (or *dof*s) of a linkage. For example, for a planar linkage with n links

[1] A plane algebraic curve of degree two.

and j pin joints or hinges, Grübler's criterion determines the degrees of freedom as $dof = 3(n - 1) - 2j$.

For a planar linkage with n links and j pin joints or hinges, Grubler's criterion determines the degrees of freedom as $dof = 3(n - 1) - 2j$.

2. Kempe's Fundamental Linkages

Below, some fundamental linkage units that Kempe conceived [1] to multiply, add or subtract angles are described. As mentioned before, these units employ exclusively parallelogram and contra-parallelogram linkages.

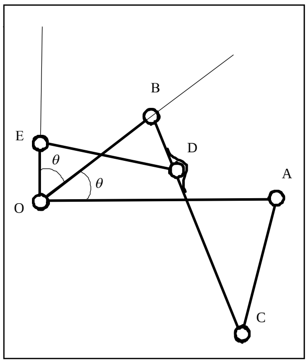
2.1 *Parallelogram and Contra-parallelogram*

While a parallelogram (linkage) is well known to many (*Figure* 1a), a contra-parallelogram might not be. In a contra-parallelogram, a vertex or a hinge P of a parallelogram is flipped as shown in *Figure* 1b. The edge lengths remain the same as in a parallelogram. Imagine points O and P to be joined to form the third side of the two triangles OBP and OAP. The lengths of OB and AP are equal to b while those of OA and BP are equal to a. Since two sides of the triangles OBP and OAP are equal and the third side OP is common, the two triangles are congruent. Therefore ∠OBP is equal to ∠OAP. Most of Kempe's basic linkages like the *reversor*, *multiplicator* and *additor* discussed next are composed of contra-parallelograms.

2.2 *The Reversor*

Let OBCA be a contra-parallelogram linkage in *Figure* 2. Thus, the length of OB (denoted by |OB|) is equal to the length of AC(i.e., |AC|), and |OA| = |BC|. Let us mark a hinge D on link BC keeping BC rigid. That is, hinges B, D and C always remain collinear irrespective of the orientation of the link BC. Let us construct another contra-parallelogram linkage OEDB such that |OE| = |BD| and |OB| = |ED|. Note that ∠OED= ∠OBC, which makes ∠OED, ∠OBC and ∠OAC equal to each other. Contra-parallelograms OEDB and OBCA are similar if the length ratio $\frac{|OE|}{|ED|}$ is equal to $\frac{|OB|}{|BC|}$. Two geometric entities are called similar if they have the

Figure 2. The Reversor.



Two geometric entities are called similar if they have the same shape but not necessarily the same size.

same shape but not necessarily the same size. In other words, in these entities, the internal angles are respectively equal and the lengths are in proportion. Since $|ED| = |OB|$ and $|BC| = |OA|$, $\frac{|OE|}{|OB|} = \frac{|OB|}{|OA|}$ implies that $|OB|^2 = |OE||OA|$. Here, $|OE|$ is said to be *third proportional* to $|OA|$ and $|OB|$.

An important consequence of the contra-parallelograms OEDB and OBCA being similar is that the included angle between EO and OB (i.e., $\angle$EOB) is the same as that between BO and OA (i.e., $\angle$BOA). Let these two angles be denoted by θ. Thus, if link OA makes an angle θ with link OB, OE makes the same angle on the other side of OB. Kempe therefore referred to this linkage as the (angle) reversor. This is true irrespective of how OA and OB are placed relative to each other. Also, note that $\angle$EOA=2θ implying that link OE always doubles the angle that the link OB makes with OA. For this reason, Kempe's reversor is also called the 'angle-doubler'.

2.3 *The Multiplicator*

If two more links OG and GF are added to the linkage in *Figure* 2 such that $\frac{|OG|}{|OE|} = \frac{|OE|}{|OB|}$, (i.e., $|OG|$ is third proportional to $|OB|$ and $|OE|$) and contra-parallelogram OGFE is constructed such that OGFE is similar to contra-parallelogram OEDB, we get the linkage shown in *Figure* 3. Then, OG makes thrice the angle that OB

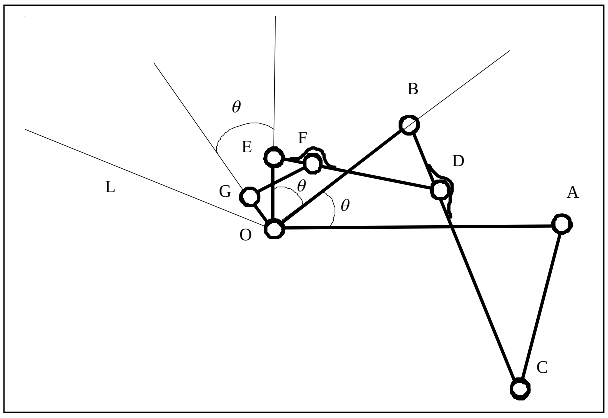


Figure 3. The Multiplicator.

makes with OA. Note that this mechanical apparatus can be conceived as an angle-trisector. If links OG and OA are aligned with two non-collinear edges, then, links OE and OB correspond to the two trisectors of the included angle ∠GOA. Trisecting an angle using straight lines (ruler) and circles (compass) alone is otherwise not possible though there exist alternative methods for it (e.g., [2]). If one continues constructing similar contra-parallelograms, then link OL will make angle $r\theta$ with OA where r is an integer. The linkage in *Figure* 3 (and the associated generalization) is called the *multiplicator* by Kempe. It is worth noting that a reversor is a special case of a multiplicator.

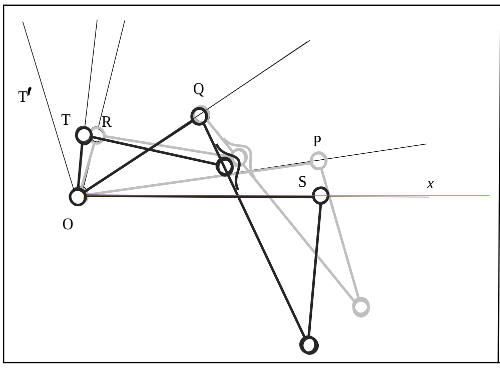
The apparatus in *Figure* 3 can be conceived as a mechanical angle trisector.

2.4 *The Additor*

Using a reversor linkage, let links OR and OP make equal angles with link OQ (*Figure* 4). By means of another reversor, let OT and OS make equal angles with OQ as well. In other words, let OQ be the common angular bisector for ∠ROP and ∠TOS. Then, since ∠ROQ = ∠QOP and ∠TOQ = ∠QOS, ∠TOR= ∠POS. Note that link OQ is common to both reversors and the hinge S lies on the x-axis.

1. For any integers m and n, if ∠POS = $m\phi$ and ∠ROS = $n\psi$, then ∠TOS = $m\phi + n\psi$.
2. If ∠POS = $m\phi$ and ∠TOS = $n\psi$, then ∠TOP = $n\psi - m\phi$ = ∠ROS.
3. If link OT′ is fixed to OT such that ∠T′OT is a fixed angle α, then ∠T′OS = $m\phi + n\psi \pm \alpha$. The sign of α depends on how OT′ is oriented relative to OT.
4. If OT′ is fixed to OR such that ∠T′ OR is α, then ∠T′OS = $n\psi - m\phi \pm \alpha$.

The linkage in *Figure* 4 is the angle additor.



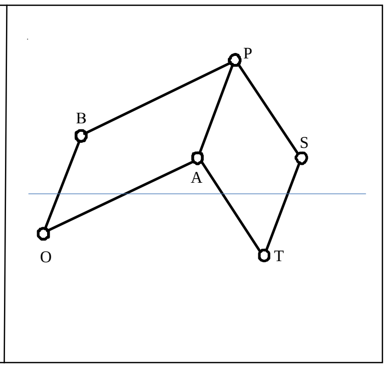


Figure 4 (left). The Additor. Figure 5 (right). The Translator.

2.5 *The Translator*

Let link OB make some angle with the horizontal (*Figure* 5). Then, via the two parallelograms OBPA and APST, link ST makes the same angle with the horizontal as do links AP and OB. The linkage unit in *Figure* 5 is the translator.

3. Kempe's Modified Fundamental Linkages

Kempe in his work [1] did not mention the cases where a parallelogram could degenerate into a straight line and/or then to a contra-parallelogram (*Figure* 6) or vice versa. Kempe implicitly assumed that parallelograms will remain parallelograms and contra-parallelograms will remain those throughout while the linkage traces the specified curve. In his concise paper of about three pages, Kempe focused on what is now known as 'Kempe's Universality Theorem' (Section 4).

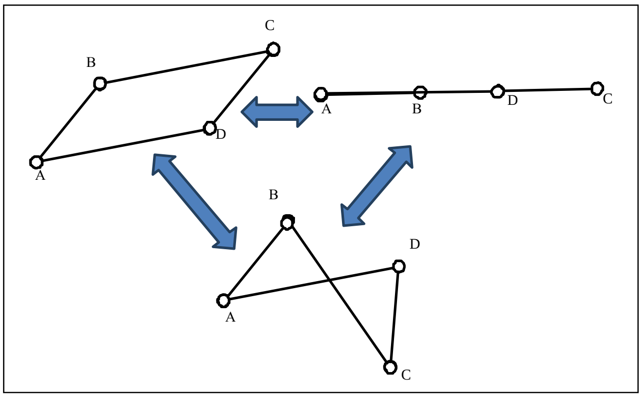


Figure 6. Possibility of a parallelogram degenerating into a straight line or a contra-parallelogram while the linkage is in motion.

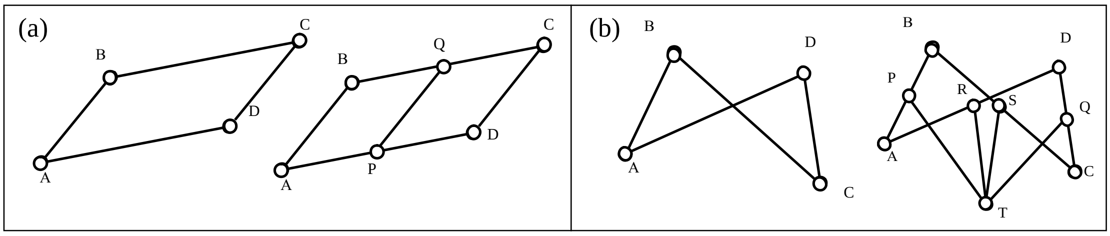


Figure 7. Bracing of the parallelogram in (a) and contra-parallelogram in (b) so that degeneration of a parallelogram into a contra-parallelogram and vice versa can be avoided.

Some modifications to the original parallelograms and contra-parallelograms were suggested in [3] so that they do not attain degenerate configurations. These modifications are shown in *Figure* 7. In *Figure* 7a (right), an extra link PQ is added such that PQ is parallel to (//) AB and DC, and |BQ| = ½|BC|. This 'bracing' does not allow a contra-parallelogram to get formed. In *Figure* 7b (right), addition of four extra links PT, RT, ST and QT is suggested. P, Q, R and S are the mid-points of AB, CD, AD and BC. |RT| = |ST| = r_1 and |PT| = |QT| = r_2. These links are large enough in length and further, $r_2^2 - r_1^2 = \frac{1}{4}(|AD|^2 - |AB|^2)$. With this bracing, degeneration of a contra-parallelogram into a parallelogram is avoided.

4. Kempe's Universality Theorem and the Construction of Kempe's Linkage

The theorem loosely states that for a polynomial $\mathcal{C}(x, y)$ of degree n, there exists a planar linkage that traces the curve $\mathcal{C}(x, y) = 0$. An example is given below that illustrates how Kempe described a thoughtful proposition of a generalized method to assemble the fundamental linkages around the parent parallelogram (*Figure* 8). Let point P be on $\mathcal{C}(x, y)$. Let OA of length l_1 make angle θ with the horizontal and OB of length l_2 make angle ψ.

For a polynomial $\mathcal{C}(x, y)$ of degree n, there exists a planar linkage that traces the curve $\mathcal{C}(x, y) = 0$.

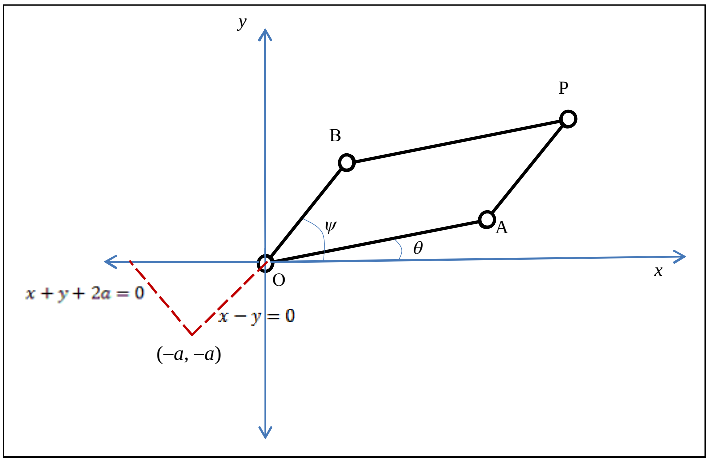
The projection of OA on the x-axis is $l_1\cos(\theta)$ and that of AP is $l_2\cos(\psi)$. Hence, the x-coordinate of P is

$$x = l_1\cos(\theta) + l_2\cos(\psi) \ . \quad (1)$$

Likewise, the y-coordinate of P is

$$y = l_1\sin(\theta) + l_2\sin(\psi) \ . \quad (2)$$

Figure 8. The starting parallelogram linkage in Kempe's construction.



Let us choose a simple curve $\mathcal{C}(x, y)$. The point P on the parallelogram is expected to trace the dotted curve shown in *Figure* 8. The curve is composed of two lines whose equations are $x - y = 0$ and $x + y + 2a = 0$ for some a, and the lines intersect at point $(-a, -a)$. Multiplication of these two equations yields

$$\mathcal{C} = (x-y)(x+y+2a) = x^2 - y^2 + 2a(x-y) = 0\,. \quad (3)$$

Note that the path has a sharp corner (a *kink*) at point $(-a, -a)$. Tracing a path having a corner can be difficult for simpler rigid-body linkages like the four bar mechanisms that have been studied extensively over the years (e.g., Hrones and Nelson Atlas [4]). Substituting (1) and (2) into (3) and using some trigonometric identities, (3) can be written in the following form.

$$1.\cos(2\theta) + 1.\cos(2\psi) + 2.\cos(\theta+\psi) + 1.\cos\left(\frac{\pi}{4}+\theta\right) + 1.\cos\left(\frac{\pi}{4}+\psi\right) = 0\,. \quad (4)$$

Equation (4) describes the summation of cosines of the angles and is obtained for $a = \frac{1}{2\sqrt{2}}$ (to make the equation look simpler) and $l_1 = l_2 = 1$ (to ensure that point P in *Figure* 8 can reach the corner in the path). The equation contains five terms and a link OL_1 to OL_5 is associated with each term. The first term in (4) requires angle θ (which OA in *Figure* 8 makes with the x-axis) to be doubled. This can be done through an angle multiplicator $\mathcal{M}1$. This doubler will be connected to link OA and will have a link OL_1 such that the angle made by it with the x-axis is 2θ (see *Figure* 2). The length of OL_1 is the coefficient of the first term in (4), that is, $|OL_1| = 1$. The second term in (4) requires angle ψ to be doubled. This can be done by another multiplicator, say $\mathcal{M}2$, that connects links OL_2 and OB (*Figure* 8) such that the angle made by OL_2 with the x-axis is 2ψ. $|OL_2| = 1$, is the coefficient of the second term. The third term requires angles θ and ψ to be added for which an additor (*Figure* 4) is needed. This additor, say $\mathcal{A}1$ will contain link OL_3 such that $|OL_3| = 2$, the coefficient of the third term, and $\angle L_3Ox = \theta + \psi$. The fourth term in (4) requires that a constant angle $\frac{\pi}{4}$ be added to θ. This can be done by attaching a rigid triangle to the link OA in *Figure* 8 whose other edge is link OL_4 such that $|OL_4| = 1$ (coefficient of the fourth term) and $\angle L_4OA = 45^o$ (or $\frac{\pi}{4}$ radians). Likewise, link OL_5 will be rigidly connected to link OB such that $\angle L_5OB = 45^o$ and $|OL_5| = 1$. Below, the geometric construction of individual linkages to locate each of the links OL_1 to OL_5 is described. Just a compass and ruler are used for the construction as is common in classical geometric constructions.

4.1 *Construction of multiplicators $\mathcal{M}1$ and $\mathcal{M}2$ to place* OL_1 *and* OL_2 *w.r.t.* OAPB

A set of links that locate OL_1 is shown in *Figure* 9. The construction steps are described below.

1. Draw the parallelogram OAPB with $|OA| = |OB| = |AP| = |BP| = 2$ units each. O, A, P and B are all hinges. OAPB is drawn such that P is on the line $x - y = 0$ for sake of convenience. In general, P can be placed arbitrarily.
2. Treat hinge O as fixed.

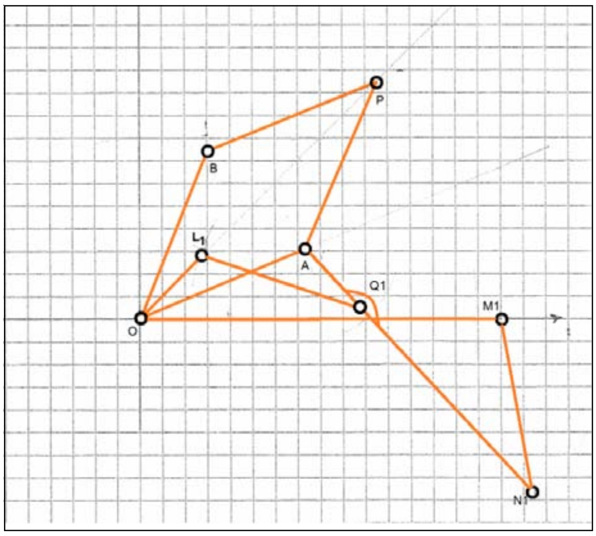


***Figure 9. Multiplicator M1 that locates the link** OL$_1$. **OL$_1$ doubles the angle made by** OA **with** OM1.*

3. Locate hinge M1 on the x-(horizontal) axis at a distance of 4 units. Hinge M1 is also fixed.

4. With M1 as the center and radius as 2 units, draw an arc (to locate hinge N1).

5. With A as center, and radius equal to 4 units, draw an arc to cut the arc in step 4.

6. N1 is the point of intersection of the two arcs.

7. Connect M1 and N1 through a link. Also, join A and N1.

8. Locate hinge Q1 on link AN1 at a distance of 1 unit from A.

9. With Q1 as center and radius as 2 units, draw an arc (to locate hinge L_1).

10. With O as center and radius as 1 unit, cut the arc in the previous step. Hinge L_1 is now determined such that $\angle L_1OM1 = 2\angle AOM1$. Create links L_1Q1 and OL_1. Link OL_1 is positioned. We now have link OL_1 corresponding to the first term in equation (4).

The linkage in *Figure* 9 has nine links (straight edges OA, AP, OB, BP, OM1, M1N1, AN1, L_1Q1 and OL_1) and 11 hinges (circles). If l links intersect at a circle, $l-1$ hinges are counted. Thus, there are three hinges at O, 2 at A and 1 each at locations B, P, L_1, Q1, M1 and N1. Link OM1 is fixed. For the mechanism in *Figure* 9, Grübler's criterion predicts its *dof* as 2 (Note: $dof = 3(n-1) - 2j$). That is, the linkage needs two independent inputs to be described completely. Since the parallelogram linkage OAPB is free to move if every element of the multiplicator M1 including the link OA is fixed, one input is needed to fix this parallelogram linkage. If, on the other hand, OB of the parallelogram is fixed, the multiplicator unit is free to operate, which is the second degree of freedom.

Steps 1–10 described above can be followed with minor changes to locate the link OL_2 (*Figure* 10). OL_2 is the link corresponding to the second term of (4). It doubles the angle ψ made by OB with the horizontal and its length is 1 unit. In the preceding steps, one needs to replace M1 by M2 (in fact they will coincide so that OM1 and OM2 is the same link), N1 by N2, A by B, and Q1 by Q2 to locate L_2 in place of L_1. The mechanism in *Figure* 10 has 9 links and 11 joints as well, implying that it can move in two possible ways just as the linkage in *Figure* 9 does.

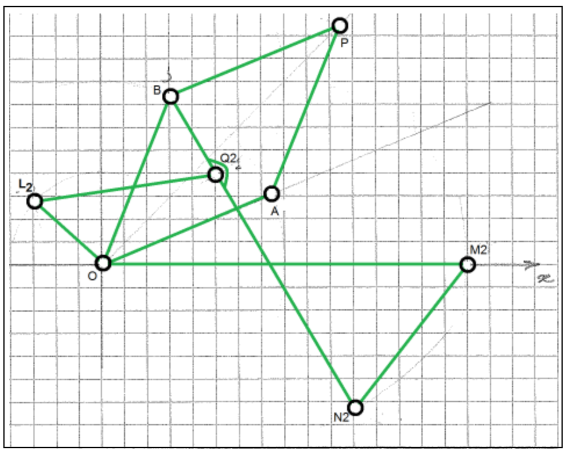


***Figure 10. Multiplicator* M2 *to locate the link* OL_2 *which doubles the angle made by* OB *with* OM2.**

4.2 *Construction of additor* $\mathcal{A}1$ *and placement of* OL_3 *w.r.t.* $\mathcal{P}$ (OAPB)

Link OL_3 is now placed (*Figure* 11) to sum the angles that links OA and OB make with the horizontal (third term in equation (4)). Also, $|OL_3| = 2$ units. The following are the construction steps for the additor unit.

1. Start with parallelogram OAPB as in Section 4.1.
2. With O as center, mark hinge M3 along OA such that $|OM3| = 4$ units.
3. Link OAM3 is a rigid link.
4. Bisect angle $\angle BOA$.
5. Locate hinge R3 on this bisector such that $|OR3| = 2$ units.
6. With R3 as center and radius 4 units, draw an arc to determine hinge N3.
7. With M3 as center and radius 2 units, cut the previous arc at N3.
8. Connect hinges R3 and N3 with a link. Also, connect N3 and M3.

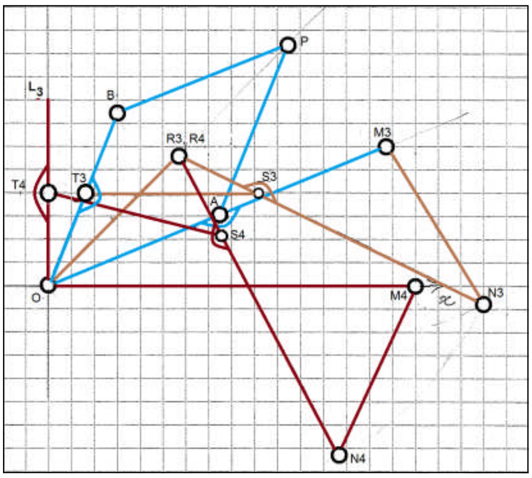


Figure 11. Orienting link OL_3 ***via Kempe's additor.*** OL_3 ***sums the angles made by links*** **OA (OM3)** ***and*** **OB** ***with link*** **OM4.**

9. Locate hinge S3 on R3N3 such that |R3S3| = 1 unit.
10. With S3 as center and radius 2 units, draw an arc to determine hinge T3. This point will lie on the link OB. Ensure that |OT3| = 1 unit. Link OT3B is a rigid link.
11. Join S3 and T3.
12. Locate hinge M4 on the x-axis such that |OM4| = 4 units.
13. With R4 as center and radius 4 units, draw an arc such that N4 lies on it.
14. R4 is the same hinge as R3, but is a part of a different reversor unit.
15. With M4 as center and radius as 2 units, cut the previous arc at hinge N4.
16. Join hinges M4 and N4, and R4 and N4.
17. Locate hinge S4 on R4N4 such that |R4S4| = 1 unit.
18. With S4 as center and radius 2 units, draw an arc such that hinge T4 lies on it.
19. With O as center and radius 1 unit, cut the previous arc at T4. Join S4 and T4.
20. Join O and T4, and extend OT4 to OL_3 such that $|OL_3|$ = 2 units and $OT4L_3$ is a single rectilinear link.

This complex mechanism has 13 links and 17 revolute pairs with link OM4 fixed. By Grübler's condition, the mechanism again has two degrees of freedom which correspond to the free motions of links OA≡OM3 and

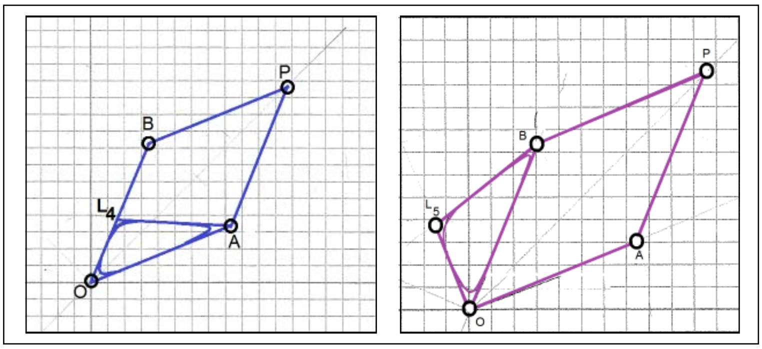


Figure 12. Rigid body triangles **OAL_4** ***(a) and*** **OAL_5** ***(b) so that*** **OL_4** ***makes 45°*** ***with*** **OA** ***and*** **OL_5** ***makes the same angle with*** **OB.**

OB. Irrespective of the angles made by OA and OB with the horizontal, OL_3 is always positioned such that $\angle L_3Ox = \angle$ AOx (which is θ) $+\angle$BOx, (which is ψ).

4.3 *Placement of* OL_4 *and* OL_5 *w.r.t.* OAPB

With the same parallelogram OAPB as in Section 4.1, extend OA to the left of O to A′. Draw the angular bisector of $\angle A'OA$. Since $\angle A'OA = 180°$, this angular bisector will be perpendicular to the segment A′OA. Let point M lie on this bisector. Bisect angle $\angle MOA$ again. L_4 lies on this new bisector at a distance of 1 unit from O so that $\angle L_4OA = 45°$. Construct line segments OL_4 and L_4A to complete the rigid triangle L_4OA. In *Figure* 12(a), points O, L_4 and B are shown to be collinear. This is because $\angle BOx$ and $\angle AOx$ are initially chosen as 67.5° and 22.5° respectively. In general, these points will not always be on a straight line. Link L_5 is located (*Figure* 12b) similar to how link L_4 is positioned in *Figure* 12 (a). Here, L_5 lies on the angular bisector of $\angle NOB$ such that $|OL_5| = 1$, where ON is perpendicular to OB. L_5OB is a rigid ternary link. Links OL_4 and OL_5 correspond to the fourth and fifth terms of (4). Both mechanisms have five links (fixed link not shown is hinged at point O) and five hinges (those at L_4 and L_5 are not accounted for since no connections exist there yet) which makes them have two degrees of freedom each as per the Grübler's criterion.

4.4 *Translation of links* OL_1 – OL_5

Links OL_1 to OL_5 are now placed such that they make angles 2θ, 2ψ, $\theta+\psi$, $45^\circ+\theta$ and $45^\circ+\psi$ respectively with the horizontal. Further, except for OL_3 whose length is 2 units, the remaining four have the lengths of 1 unit each. Equation (4) in fact describes the sum of the distances of hinges L_1 to L_5 from the Oy-axis. To facilitate this, one now needs to make a serial chain of links such that the tail of OL_i coincides with the tip of link OL_{i-1}. That is, the tail of link OL_2 coincides with the tip of link OL_1 and so on until the tail of OL_5 meets with the tip of OL_4. This is accomplished via the use of translators (see section 2.5). Consider *Figure* 13 which shows links OL_1 to OL_5 obtained from the previous constructions. The parallelogram OAPB is not depicted. Link OL_2 is translated parallel to itself such that its tail lies over the tip of OL_1 (*Figure* 13). This is done via the parallelogram $OL_1L_2^PL_2$. Link OL_3 is translated first to lie over OL_1 and then over OL_2. This is done via two translators. The final position of OL_3 is $L_2^PL_3^P$. Likewise OL_4 is translated via three parallelograms to its final position $L_3^PL_4^P$ and OL_5 via four translators to $L_4^PL_5^P$. In this linkage, one can find that the net number of links is 26 and those of the joints is 35. Thus, the total degrees of freedom are 5 which is expected since all links OL_1, OL_2, ..., OL_5 are free to rotate about O.

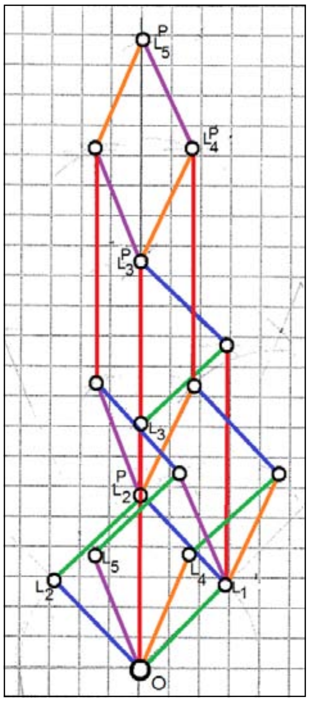


Figure 13. Links OL_i translated to get the extreme point L^P_5. The parent parallelogram OAPB is not shown.

The assembly of the final Kempe's mechanism is depicted in *Figure* 14 wherein all linkages (*Figures* 9–13) are placed together. The Peaucellier–Lipkin cell is placed additionally to guide L_5^P along the vertical line (Oy) by rotating the link MN. Topologically, the mechanism is far from being simple. To compute the total number of links and joints, consider the following assembly procedure.

The additor linkage is placed first – this contributes 13 links and 17 hinges. Next, multiplicators M1 and M2 are positioned. Note that the parallelogram OAPB and the fixed links OM1 ≡ OM2 ≡ OM4 are common between the three mechanisms (*Figures* 9–11). To avoid

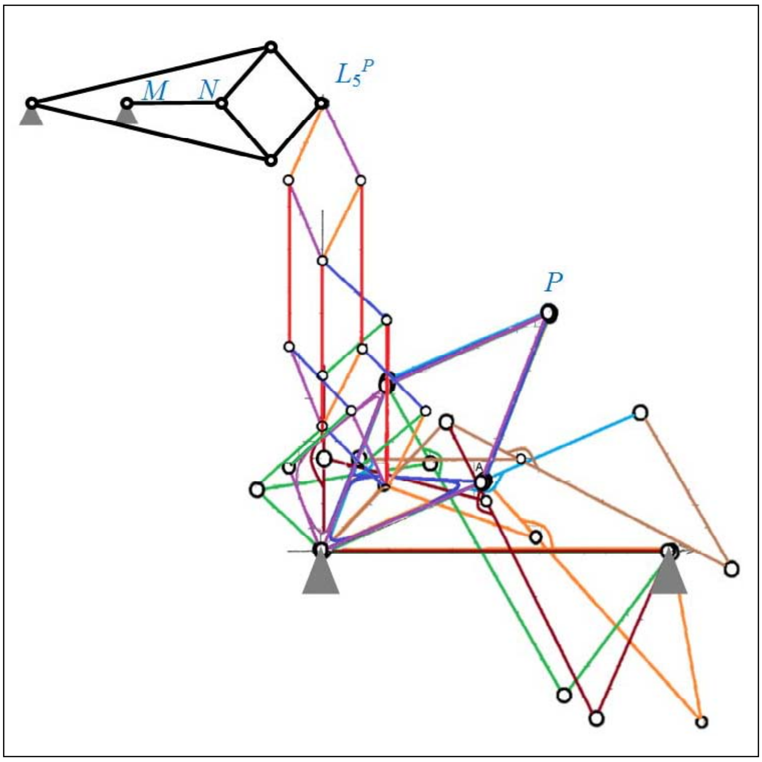


Figure 14. Kempe's linkage. The final assembly that traces the curve **(x–y) (x+y+1/$\sqrt{2}$) = 0.**

counting some links and hinges more than once, from M1 and M2 each, only 4 out of 9 links and 6 out of 11 hinges which are not common are considered. Linkages in *Figure* 12 do not contribute any additional link or a hinge. This is because the parallelograms are common and triangles L_4OA and L_5OB are rigid extensions of OA and OB respectively. When considering the chain of parallelograms (*Figure* 13), six links and five hinges are discarded from the original respective count of 26 and 35. Links OL_1, OL_2, ..., OL_5 and the fixed link are already parts of the previous five linkages. For a similar reason, all hinge joints at location O in *Figure* 13 are ignored. However, hinges at locations L_1, L_2, ..., L_5 are retained because at these joints, connections between the linkage in *Figure* 13 and those in *Figures* 9–12 are made. Thus the chain of parallelograms in *Figure* 13 contributes 20 links and 30 joints. Ignoring the

Peaucellier–Lipkin mechanism, the total number of links is 41 while the net number of joints is 59 for which the Grübler's criterion gives the net degrees of freedom as 2. Now, if one considers this cell and discounts the fixed link that is already considered before, the numbers of links and joints get revised to 48 and 70. Grübler's criterion prediction for the total degrees of freedom of the entire Kempe's linkage is 3(47) – 2(70) = 1. This suggests that if point L_5^P is moved along the y-axis, point P has to trace the curve $(x-y)(x+y+\frac{1}{\sqrt{2}})=0$ for which the Kempe's linkage is designed.

5. Closure

What Kempe suggested more than a hundred years ago was to allow for adequate degrees of freedom in a path-generating linkage. Kempe definitely knew and shared through his analytical method how to do so with rigid links. He did not have access to the computational methods and facilities we enjoy today. But he had far more insight into mechanism design. Analytical methods, like the one proposed by Kempe, for the design of the simplest linkages still elude us despite the extensive computational power we have today!.

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Suggested Reading

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