An Intuitive Solution of a Convexity Problem

In this article we provide a solution to a problem in the famous analysis book [1] by Rudin. It does not use transfinite induction, and readers may find it more transparent than the treatment in [2]. Here is the statement of the problem.

Assume that $f$ is a continuous real valued function defined in $(a,b)$ such that

$$f \left( \frac{x + y}{2} \right) \leq \frac{f(x) + f(y)}{2} \leq \forall x, y \in (a,b);$$

then $f$ is convex on $(a,b)$ (Question 24, p.101 in [1]).

**Solution:** Assume that $f$ is continuous and satisfies the conditions stated in the problem. If possible, suppose that $f$ is not convex on $(a,b)$. Then there exists some $\lambda$ in $(0,1)$ and $x_1, x_2$ in $(a,b)$, $x_1 < x_2$, such that

$$f(\lambda x_1 + (1 - \lambda)x_2) > \lambda f(x_1) + (1 - \lambda)f(x_2),$$

that is

$$f(c) > \lambda f(x_1) + (1 - \lambda)f(x_2),$$

where $c = \lambda x_1 + (1 - \lambda)x_2$. This means that the point $[c, f(c)]$ lies above the segment $PQ = \{\lambda[x_1, f(x_1)] + (1 - \lambda)[x_2, f(x_2)] : 0 \leq \lambda \leq 1\}$ joining the points $P[x_1, f(x_1)]$ and $Q[x_2, f(x_2)]$.

For brevity, we say ‘$f(x)$ lies on segment $PQ$’ to mean that the point $[x, f(x)]$ lies on segment $PQ$. Define

$$G = \{y \in [x_1, c] : f(y) \text{ lies on segment } PQ\},$$

and

$$H = \{y \in (c, x_2] : f(y) \text{ lies on segment } PQ\};$$

**Keywords**

Convexity, convex function, midpoint domination condition, transfinite induction.
then $G$ and $H$ both are nonempty and bounded.

**Claim**

$G$ and $H$ are closed.

**Proof**

Let $y_n \in G$ be a sequence converging to $y$ in $(a, b)$. As $f$ is continuous, the sequence $f(y_n)$ converges to $f(y)$. Since $f(y_n)$ lies on segment $PQ$ for each $n$, $f(y)$ also lies on segment $PQ$. Hence $G$ is closed. Similarly for $H$.

Hence $G$ and $H$ both are nonempty compact sets. Define $\max G = g$ and $\min H = h$. Clearly $g < c < h$. By construction, the points $[z, f(z)]$ lie above the line segment connecting $[g, f(g)]$ and $[h, f(h)]$, for all $z$ in the interval $(g, h)$. Since $\frac{1}{2}(g + h)$ lies in $(g, h)$, this implies in particular that

$$f\left(\frac{g + h}{2}\right) > \frac{f(g) + f(h)}{2}.$$  

This contradicts the mid-point domination condition. Hence $f$ is convex.

**Suggested Reading**


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