Triloki Nath and S R Singh
Department of Mathematics,
Banaras Hindu University
Varanasi 221005, India.
Email: tnverma07@gmail.com
Email:
singh_shivaraj@rediffmail.com

An Intutive Solution of a Convexity Problem

In this article we provide a solution to a problem in the famous analysis book [1] by Rudin. It does not use transfinite induction, and readers may find it more transparent than the treatment in [2]. Here is the statement of the problem.

> *Assume that f is a continuous real valued function defined in (a,b) such that*
>
> $$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \quad \leq \ \forall\ x,y \in (a,b);$$
>
> *then f is convex on (a,b)* (Question 24, p.101 in [1]).

Solution: Assume that f is continuous and satisfies the conditions stated in the problem. If possible, suppose that f is not convex on (a,b). Then there exists some λ in $(0,1)$ and x_1, x_2 in (a,b), $x_1 < x_2$, such that

$$f\left(\lambda x_1 + (1-\lambda)x_2\right) > \lambda f(x_1) + (1-\lambda) f(x_2),$$

that is

$$f(c) > \lambda f(x_1) + (1-\lambda) f(x_2),$$

where $c = \lambda x_1 + (1-\lambda)x_2$. This means that the point $[c, f(c)]$ lies above the segment $PQ = \{\lambda[x_1, f(x_1)] +(1-\lambda)[x_2, f(x_2)] \ :\ 0 \leq \lambda \leq 1\}$ joining the points $P[x_1, f(x_1)]$ and $Q[x_2, f(x_2)]$.

For brevity, we say '$f(x)$ lies on segment PQ' to mean that the point $[x, f(x)]$ lies on segment PQ. Define

$$G = \left\{y \in \left[x_1, c\right) : f(y) \text{ lies on segment } PQ\right\},$$

and

$$H = \left\{y \in \left(c, x_2\right] : f(y) \text{ lies on segment } PQ\right\};$$

Keywords
Convexity, convex function, mid-point domination condition, transfinite induction.

then G and H both are nonempty and bounded.

Claim

G and H are closed.

Proof

Let $y_n \in G$ be a sequence converging to y in (a, b). As f is continuous, the sequence $f(y_n)$ converges to $f(y)$. Since $f(y_n)$ lies on segment PQ for each n, $f(y)$ also lies on segment PQ. Hence G is closed. Similarly for H.

Hence G and H both are nonempty compact sets. Define $\max G = g$ and $\min H = h$. Clearly $g < c < h$. By construction, the points $[z, f(z)]$ lie above the line segment connecting $[g, f(g)]$ and $[h, f(h)]$, for all z in the interval (g, h). Since $\frac{1}{2}(g + h)$ lies in (g, h), this implies in particular that

$$f\left(\frac{g+h}{2}\right) > \frac{f(g) + f(h)}{2}.$$

This contradicts the mid-point domination condition. Hence f is convex.

Acknowledgment

We thank S Bhattacharya, Department of Mathematics, Banaras Hindu University, Varanasi, India, for valuable suggestions to prepare this article.

Suggested Reading

[1] W Rudin, *Principles of mathematical analysis*, International Series in Pure and Applied Mathematics, McGraw-Hill Book Co., New York-Auckland-Düsseldorf, 1976.

[2] K G Binmore, *Mathematical analysis*. A straight forward approach. Cambridge University Press, Cambridge-New York-Melbourne, 1977.