Think It Over

This section of *Resonance* presents thought-provoking questions, and discusses answers a few months later. Readers are invited to send new questions, solutions to old ones and comments, to 'Think It Over', *Resonance*, Indian Academy of Sciences, Bangalore 560 080. Items illustrating ideas and concepts will generally be chosen.

Solution to Motion with a Constraint

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Problem 1. Consider a walker *W* who walks in a big playground such that *W* is always twice as far from a point A_1, as *W* is from a point A_2. Here A_1 and A_2 are fixed. Find the path traced by the walker *W*.

Problem 2. Consider a bird *B* that is flying in space in such a way that it is always two times as far from the top of tower T_1 as it is from the top of tower T_2. What is the collection of points in space traced by the bird *B*?

Problem 3. Formulate and solve a generalization of the above two problems to higher dimensions.

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1. The playground can be identified with a subset of the (X, Y) plane and the line through A_1 and A_2 as the X- axis. Let the coordinates of A_1 be $(-a, 0)$, and A_2 be $(a, 0)$. So the distance between A_1 and A_2 is $2a$ and (0,0) is the mid-point of the segment $\mathrm{A}_1\mathrm{A}_2$ along the X-axis. Let $W = (x, y)$ be a position of the walker. The given condition is that $d(W, \mathrm{A}_i) \equiv$ distance of W to A_i, $i = 1, 2$ satisfy

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$$d(W, \mathrm{A}_1) = 2d(W, \mathrm{A}_2). \quad (1)$$

Thus

$$d^2(W, \mathrm{A}_1) = 4d^2(W, \mathrm{A}_2).$$

But

$$\begin{aligned} d^2(W, \mathrm{A}_1) &= (x+a)^2 + y^2 \quad \text{and} \\ d^2(W, \mathrm{A}_2) &= (x-a)^2 + y^2. \end{aligned} \quad (2)$$

So (x, y) satisfies

$$(x+a)^2 + y^2 = 4((x-a)^2 + y^2). \quad (3)$$

That is

$$x^2 + 2ax + a^2 + y^2 = 4(x^2 - 2ax + a^2 + y^2).$$

This implies

$$3x^2 - 10ax + 3a^2 + 3y^2 = 0,$$

i.e.,

$$3\left(x - \frac{5a}{3}\right)^2 + 3y^2 = \left(\frac{3 \times 25}{9} - 3\right)a^2,$$

i.e.,

$$\left(x - \frac{5a}{3}\right)^2 + y^2 = \left(\frac{16}{9}\right)a^2. \quad (4)$$

Conversely, if (x, y) satisfies (4) it satisfies (1). Thus the path traced by the walker is the set of points $\{W : d(W, C) = r\}$, where

$$C = \left(\frac{5a}{3}, 0\right), \quad r = \frac{4}{3}a.$$

That is, the path traced by W is a circle with center at $\left(\frac{5a}{3}, 0\right)$ and radius $\frac{4}{3}a$. Note that the center of this circle is in the segment $\mathrm{A}_1\mathrm{A}_2$ on the X-axis. Its distance to A_1 is $\frac{8a}{3}$ and to A_2 is $\frac{2a}{3}$, and the points $W_1 \equiv \left(\frac{a}{3}, 0\right)$, $W_2 = (3a, 0)$ both lie on this circle and $d(W_1, \mathrm{A}_1) = \frac{4a}{3} =$

$2d(W_1, \mathrm{A}_2) = 2\times\frac{2a}{3}$ and $d(W_2, \mathrm{A}_1) = 4a = 2d(W_2, \mathrm{A}_2) = 2 \times 2a$.

2. As in the solution to Problem 1 above but $\mathrm{T}_i = (a_i, b_i, c_i), i = 1, 2$ and $B = (x, y, z)$. The given constraint can be expressed as

$$(x - a_1)^2 + (y - b_1)^2 + (z - c_1)^2 =$$

$$4((x - a_2^2)^2 + (y - b_2)^2 + (z - c_2)^2).$$

Simplifying this yields

$$3\left(x^2 - \left(\frac{(8a_2 - 2a_1)}{3}\right)x\right) + 3\left(y^2 - \left(\frac{8b_2 - 2b_1}{3}\right)y\right)$$

$$+3\left(z^2 - \left(\frac{8c_2 - 2c_1}{3}\right)z\right) = 4(a_2^2+b_2^2+c_2^2)-(a_1^2+b_1^2+c_1^2).$$

This is equivalent to the condition

$$(x - d_1)^2 + (y - d_2)^2 + (z - d_2)^2 = \alpha, \qquad (5)$$

where $d_1 = \frac{8a_2-2a_1}{3\times2}$, $d_2 = \frac{8b_2-2b_1}{3\times2}$, $d_3 = \frac{8c_2-2c_1}{3\times2}$, and

$$\alpha = \frac{4}{3}(a_2^2 + b_2^2 + c_2^2) - \frac{1}{3}(a_1^2 + b_1^2 + c_1^2) + d_1^2 + d_2^2 + d_3^2.$$

Since the left-side of (5) is ≥ 0, α has to be nonnegative and so can be written as r^2. Further α is a constant independent of (x, y, z).

Let $D \equiv (d_1, d_2, d_3)$. Then $B = (x, y, z)$ is a solution to the constraint $d(B, \mathrm{T}_1) = 2d(B, \mathrm{T}_2)$ iff

$$d(B, D) = r.$$

That is, the orbit of the bird B is the surface of a sphere with center D and radius r.

3. (Generalization to n-space) Given two points A_1 and A_2 in an n-dimensional Euclidean space R_n the set of points P in R_n that satisfy

$$d(P, \mathrm{A}_1) = \lambda d(P, \mathrm{A}_2), \quad \text{where} \quad \lambda > 0, \lambda \neq 1,$$

is the surface of a sphere in R_n with center at C and radius r, where C and r can be determined by the coordinates of A_1, A_2 and λ. If $\lambda = 1$ than the path is the perpendicular bisector of the line segment A_1, A_2, i.e., an $(n-1)$-dimensional space that goes through the mid-point of $\mathrm{A}_1\mathrm{A}_2$ and is normal to $\mathrm{A}_1\mathrm{A}_2$.

4. Further Generalization

An inner product space (IPS) is a set V with three operations defined as follows:

(i) For any two elements v_1 and v_2 in V there is an element v_3 called their sum $(v_1 + v_2)$ in V and $(v_1 + v_2) = (v_2 + v_1)$.

(ii) For any real number α and element v in V there is an element αv in V.

(iii) For any two elements v_1, v_2 in V there is a real number called $\langle v_1, v_2\rangle$ such that $\langle v_1, v_2\rangle = \langle v_2, v_1\rangle$.

Condition (i) is called vector addition, (ii) scalar multiplication and (iii) an inner product provided it satisfies the following:

(a) V under (i) and (ii) is a *vector space* and (iii) satisfies

(b)

$$\begin{aligned}
&\langle v_1, v_2\rangle = \langle v_2, v_1\rangle \\
&\langle \alpha_1 v_1 + \alpha_2 v_2 \alpha_3 v_3\rangle = \alpha_1\alpha_3\langle v_1, v_3\rangle + \alpha_2\alpha_3\langle v_2, v_3\rangle \\
&\langle v, v\rangle \geq 0, \quad \text{and } = 0 \quad \text{iff} \quad v = 0.
\end{aligned}$$

Now define a distance between elements of V by the formula

$$d(v_1, v_2) = \sqrt{\langle v_1 - v_2, v_1 - v_2\rangle}.$$

One can verify that $d(.,.)$ satisfies

(i) $d(v_1, v_2) = d(v_2, v_1) \geq 0, = 0$ iff $v_1 = v_2$

(ii) $d(v_1, v_3) \leq d(v_1, v_2) + d(v_2, v_3)$.

Now fix two vectors v_1 and v_2 in V and a positive number $0 < \lambda < \infty$. Let S be the set of vectors v in V such that $d(v, v_1) = \lambda d(v, v_2)$.

Then, if $\lambda = 1$, the set S consists of all vectors V such that

$$\left\langle v - \frac{v_1 + v_2}{2}, \quad v_1 \right\rangle = 0$$

$$\left\langle v - \frac{v_1 + v_2}{2}, \quad v_2 \right\rangle = 0$$

If $\lambda \neq 1$ then there exists a vector v_0 and a positive number such that $S = \{v : d(v, v_0) = r\}$.

To see this note that

$$\begin{aligned}
& d(v, v_1) = \lambda d(v, v_2) \\
\Leftrightarrow \quad & \langle v - v_1, v - v_1 \rangle = \lambda^2 \langle v - v_2, v - v_2 \rangle \text{ , i.e.,} \\
& \langle v, v \rangle (1 - \lambda^2) - 2\langle v, v_1 \rangle + \langle v_1, v_1 \rangle = \\
& -2\lambda^2 \langle v, v_2 \rangle + \lambda^2 \langle v_2, v_2 \rangle \\
\Leftrightarrow \quad & \langle v, v \rangle - 2 \left\langle v, \frac{v_1 - \lambda^2 v_2}{1 - \lambda^2} \right\rangle \\
& = \frac{\lambda^2 \langle v_2, v_2 \rangle - \langle v_1, v_1 \rangle (1 - \lambda^2)}{1 - \lambda^2} \\
\Leftrightarrow \quad & \left\langle v - \frac{v_1 - \lambda^2 v_2}{1 - \lambda^2}, \quad v - \frac{v_1 - \lambda^2 v_2}{1 - \lambda^2} \right\rangle \\
& = \frac{\lambda^2 \langle v_2, v_2 \rangle - \langle v_1, v_1 \rangle (1 - \lambda^2)}{(1 - \lambda^2)} + \\
& \frac{\langle v_1 - \lambda^2 v_2, v_1 - \lambda^2 v_1 \rangle}{(1 - \lambda^2)^2} \\
\Leftrightarrow \quad & d\left(v, \frac{v_1 - \lambda^2 v_2}{1 - \lambda^2}\right) = r
\end{aligned}$$

where r is nonnegative such that r^2 equal the right-side above.