Extending Given Digits to make Primes or Perfect Powers

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A question which can be asked by anyone even at the middle school level is whether a given string of digits could be extended to obtain primes or other special types of numbers. Here, answers are given which may not be accessible to a student at that level but should be accessible at least to undergraduate students. The treatment is elementary but hints to deeper mathematics connected to such elementary problems.

B Sury was associated with *Resonance* during 1999–2005. He introduces this note with:

Any sequence of digits can be amended by adding a tail and extended to get a power – more or less – of whatever and maybe even a prime number!

1. Introduction

Start with any string of digits. Can we always put down some more digits on the right of it to get a prime? Can we similarly get a power of 2? How about a power of 3? It turns out that the answers to all these questions are in the affirmative. Our discussion will be elementary excepting a concrete consequence of a weak version of the prime number theorem. For a really detailed analysis of the proportion of primes with given starting digits, the interested reader is referred to look at ergodic theory. There is no prime-producing polynomial in a single variable – this is trivial to see. However, it turns out (as other concrete consequences of the properties of primes like Bertrand's postulate and the prime number theorem) that there are exponential type of functions which produce infinitely many primes. One such is the sequence of integer parts of t^{3^n} for a certain positive real number t. Another is the sequence of integer parts of $2^{2^{\cdot^{\cdot^{s}}}}$ for a certain real $s > 0$. We shall prove these also.

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2. Perfect Powers with a Given Beginning

Let us first deal with the problem of extending a given string of digits to make a power of any natural number a which is at least 2 but not a power of 10. Notice that these exceptions are clearly unavoidable; there is no way to start with, say 11, and get a power of 10 by adding any number of digits. Let $a > 1$ be any natural number other than a power of 10. Let A be any given natural number in base 10. The only property we need is the following observation which can be proved simply by using the pigeon-hole[1] principle. For any real number α, let us write $\{\alpha\}$ for the fractional part of α.

Observation. For any irrational number $\theta > 0$, the sequence of fractional parts $\{n\theta\}$ as n varies over natural numbers, is dense in the interval $(0, 1)$.

Proof. For any n, consider the intervals $[0, 1/n), [1/n, 2/n), \cdots, [(n-1)/n, 1)$. By the pigeon-hole principle, among the fractional parts of $\theta, 2\theta, \cdots, n\theta, (n+1)\theta$, there must be at least two (say $r\theta, s\theta$ with $r < s \leq n+1$) which are in the same interval $[(k-1)/n, k/n)$. But then the fractional part of $(s-r)\theta$ lies in $[0, 1/n)$. Therefore, each $[(m-1)/n, m/n]$ contains the fractional part of some $d\theta$, where d is a multiple of $s-r$. As every subinterval (x, y) of $(0, 1)$ contains an interval of the form $[(m-1)/n, m/n]$ for some m, n, the claim asserted follows.

Using this, one may extend any given digits to produce powers as follows.

Lemma 1. Let $a > 1$ be not a power of 10 and let A be any given natural number. Then, one may add digits to the right end of the digits of A to obtain some power of a.

Proof. For any a as above, $\log_{10}(a)$ is irrational because, if it is u/v, then $10^u = a^v$ which implies by uniqueness of prime decomposition that a must be a power of 10,

[1]One of the basic principles of counting is the 'pigeon-hole principle'. It asserts that if n objects are put into m pigeon-holes with m less than n, then at least one pigeon-hole must contain more than one object.
Dirichlet is said to have first formulated it as *Schubfachprinzip*' (shelf-principle) in 1834. It turns out to be unexpectedly powerful in mathematics as well as in computer science.
For instance, one can deduce from the pigeon-hole principle that in a gathering of people there is always a pair of people who know the same number of people.
Another implication is that given any sequence of n integers, one can choose some whose sum is a multiple of n.

a contradiction of the hypothesis. So, it follows by the above observation that each interval $(x, y) \subseteq (0, 1)$ contains some fractional part $\{n \log_{10}(a)\}$. Now, suppose A has $d+1$ digits; that is, $10^d \leq A < 10^{d+1}$. Then, we consider $x \in (0, 1)$ such that $10^x = A/10^d$. Choosing some large n so that $10^{1/n} < 1 + \frac{1}{A}$ (as $\lim_{n\to\infty} 10^{1/n} = 1$), we consider the point $y \in (x, 1)$ with $10^y = \frac{10^{1/n}A}{10^d}$. Note that $y < 1$ because $10^{1/n}A < A + 1 \leq 10^{d+1}$. If the fractional part $\{r \log_{10}(a)\} \in (x, y)$, we have

$$x < r \log_{10}(a) - k < y$$

for some positive integer k. Taking 10-th powers, we have

$$10^x < \frac{a^r}{10^k} < 10^y$$

which gives

$$10^{k-d}A < a^r < 10^{k-d}10^{1/n}A < 10^{k-d}A + 10^{k-d}.$$

Thus, a^r has been obtained by adding $k-d$ digits to the right of the base 10 expansion of A.

Illustration. Let us see how to demonstrate the above lemma for a small number. Let us begin with $A = 4$ and $a = 2$. Of course A itself is a power of 2 but let us see what we get from the above lemma. In the above notation, $d = 0$ and $x = \log_{10}(4)$. The choice $n = 16$ is large enough so that $10^{1/n} < 1 + 1/4 = 1.25$. Then $y = x + 1/16$ and the choice of r, k such that

$$10^x = 4 < \frac{2^r}{10^k} < 10^y = 4 + \frac{1}{16}$$

can be taken to be $r = 12, k = 3$. Hence 4 can be extended to $2^{12} = 4096$.

3. Primes with a Given Beginning

Now, we consider the problem of extending given digits on the right to get a prime. Here, we will need the

There exists n_0 so that for $n \geq n_0$, there is always a prime strictly between n and $n + \frac{n}{\log n}$.

following property of prime numbers which is a weak consequence of the so-called prime number theorem:

There exists n_0 *so that for* $n \geq n_0$, *there is always a prime strictly between* n *and* $n + \frac{n}{\log n}$.

Lemma 2. Let A *be any given natural number. Then, one may add digits to the right end of the digits of* A *to obtain a prime number.*

Proof. As our purpose is to add digits to the right end, we may assume that $A \geq n_0$, where n_0 is as above. Now, let $r > \frac{A}{\log_e(10)}$ and consider a prime p between $10^r A$ and $10^r A + \frac{10^r A}{\log_e(10^r A)}$. Thus, $p = 10^r A + d$, where $d < \frac{10^r A}{\log_e(10^r A)} < 10^r$ since $A < \log_e(10^r) = r \log_e(10)$ by the choice of r.

Finally, we note that the last lemma implies the following:

COROLLARY 1

The fractional parts of $\log_{10}(p)$, *as* p *runs over primes, is dense in* $(0, 1)$.

Proof. Let n be arbitrary and divide $(0, 1)$ into the intervals $(0, 1/n), [1/n, 2/n), \cdots, [(n-1)/n, 1)$. Consider the numbers $10^{(m-1)/n}, 10^{m/n} \in [1, 10)$. By Lemma 2, for each $m < n$, there is a prime p and some integer d so that

$$10^{(m-1)/n} < \frac{p}{10^d} < 10^{m/n}.$$

Thus, $(m-1)/n < \log_{10}(p/10^k) < m/n$, which means that the fractional part of $\log_{10}(p)$ lies in $((m-1)/n, m/n)$. This completes the proof of the corollary as m, n are arbitrary. 3mm

The fractional parts of $\log_{10}(p)$, as p runs over primes, is dense in (0,1) but is not uniformly distributed.

We remark that using a weak version of the prime number theorem for arithmetic progressions, one may similarly prove that given any string of beginning digits and any string of end digits which end in $1, 3, 7$ or 9, one may introduce digits in between to get a prime.

Using Bertrand's postulate, one can produce a sequence of primes as the integer parts of $2^{2^{s}}$ for a certain real s > 0.

4. Some Exponential Functions Producing Primes

The so-called Bertrand postulate (see [1] for instance) tells us that there is a prime between N and $2N$ for each $N > 1$. Using this, one can write down a function which produces infinitely many primes. Let us first discuss this 1951 result due to E M Wright [2].

Lemma 3. There exists a real number $s > 0$ such that the sequence $a_0 = s, a_1 = 2^s, a_2 = 2^{2^s}, \cdots, a_{n+1} = 2^{a_n}$ produces primes $[a_n] = [2^{2^{2^{\cdot^{\cdot^{s}}}}}]$ for all $n > 0$.

Proof. Let $p_1 = 2, p_2 = 3$ and choose primes p_n for $n > 2$ such that

$$2^{p_n} < p_{n+1} < p_{n+1} + 1 < 2^{p_n+1}.$$

Look at the sequences b_n and c_n defined as follows: $b_n = \log_2 \log_2 \cdots \log_2(p_n)$ and $c_n = \log_2 \log_2 \cdots \log_2(p_n + 1)$, where there are n logarithms to the base 2. Then, we have

$$p_n < \log_2 p_{n+1} < \log_2(p_{n+1} + 1) < p_n + 1 \ .$$

This means $b_n < b_{n+1} < c_{n+1} < c_n$ which ensures that the sequence $\{b_n\}$ converges to some real number s as $n \to \infty$. Notice that for this number s, the sequence $a_n = 2^{2^{2^{\cdot^{\cdot^{s}}}}}$ satisfies $p_n < a_n < p_n + 1$. Hence $p_n = [a_n]$. This completes the proof.

Remarks. (i) The above formula is not a practical one. Since there is a choice of p_n's allowed, the real number s is not unique. One possible value of s is $1.9287800\cdots$ and the primes p_n defined by Lemma 3 grow much too fast. For example, p_4 has 5000 digits.

(ii) A result earlier to Wright's result above (in fact, the result which motivated Wright's theorem) is due to W H Mills [3] in 1947. This uses a result on primes which

There is a positive number c such that $p_n + c\,p_n^{5/8} > p_{n+1}$ for all *n*, where $p_1<p_2<p_3< \ldots$ is the sequence of all primes.

is considerably deeper than Bertrand's postulate. This deeper result alluded to is due to the British mathematician A E Ingham; he derived in 1937 (see [4]) the following concrete consequence of the prime number theorem:

There is a positive number c such that $p_n + cp_n^{5/8} > p_{n+1}$ for all n, where $p_1 < p_2 < p_3 < \cdots$ is the sequence of all primes.

Lemma 4. There exists a real number $t > 0$ such that $[t^{3^n}]$ is a prime for every n.

Proof. Start with Ingham's result and choose a large $N > c^8$. Look at the prime p_n such that $p_n < N^3 < p_{n+1}$. Then, we have

$$p_n < N^3 < p_{n+1} < p_n + cp_n^{5/8} < N^3 + cN^{15/8} < N^3 + N^2$$
$$< (N+1)^3 - 1.$$

Take for N, a prime $p > c^8$. Thus, we have a sequence of primes $p_{r_0} = p < p_{r_1} < p_{r_2} < \cdots$ such that

$$p_{r_n}^3 < p_{r_{n+1}} < (p_{r_n} + 1)^3 - 1 \qquad (1)$$

Then the sequences $u_n = p_{r_n}^{3^{-n}}$ and $v_n = (p_{r_n} + 1)^{3^{-n}}$ satisfy

$$v_n = (p_{r_n} + 1)^{3^{-n}} > (p_{r_{n+1}} + 1)^{3^{-n-1}} = v_{n+1} > p_{r_{n+1}}^{3^{-n-1}}$$
$$= u_{n+1} > p_{r_n}^{3^{-n}} = u_n.$$

Indeed, the inequality $v_n > v_{n+1}$ is simply the second inequality in (1); the inequality $u_{n+1} > u_n$ is the first inequality of (1) and the inequality $v_{n+1} > u_{n+1}$ is obvious. Hence, the sequence $\{u_n\}$ is a bounded, monotonically increasing sequence and must have a limit t. Clearly, $t = \lim_{n\to\infty} u_n$ satisfies $u_n \quad t < v_n$. Thus,

$$p_{r_n} \quad t^{3^n} < p_{r_n} + 1.$$

This proves that $[t^{3^n}] = p_{r_n}$ for all n.

Mills's existential result can be made effective using the Riemann hypothesis.

Remarks on Mills' Constant. Mills proved only the existence of a constant t as above. Later, others showed that there are uncountably many choices for t but it is still not possible to produce a value of t which can be proven. Under the Riemann hypothesis, one can prove that there is a value of t which is between 1.3 and 1.31 for which the sequence $[t^{3^n}]$ gives primes.

Suggested Reading

[1] B Sury, How far apart are primes – Bertrand's postulate, *Resonance*, Vol.7, No.6, pp.77–87, 2002.

[2] E M Wright, A prime-representing function, *Amer. Math. Monthly*, Vol.58, No.9, pp.616–618, 1951.

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[4] A E Ingham, On the difference between consecutive primes, *Quart.J.Math.*, (Oxford Ser), Vol.8, pp.255–266, 1937.

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