Classroom

In this section of *Resonance,* we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

On Pythagorean Triples of the form (i, i + 1, k)

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A Pythagorean triple is a triad of positive integers (x, y, z) which satisfy the Pythagoras' equation $x^2 + y^2 = z^2$. In this article, we shall consider triples of the form (i, $i + 1$, k), and the recurrence relations governing them. In the process, we also solve completely the equation $i^2 + (i + 1)^2 = k^2$.

1. Introduction

A Pythagorean triple consists of three positive integers a, b, c such that $a^2 + b^2 = c^2$. Such a triple is denoted by (a, b, c). A common example is $(3, 4, 5)$. If (a, b, c) is a Pythagorean triple, then so is (ta, tb, tc) for any positive integer t. A triple (a, b, c) is called a *primitive* pythagorean triple if a, b, c are mutually coprime.

The name is derived from the Pythagorean Theorem, which states that every right triangle has side-lengths satisfying the formula $a^2 + b^2 = c^2$; thus, Pythagorean triples describe the three integer side-lengths of a right triangle. However, right triangles with non-integer sides do not form Pythagorean triples. For instance, the triangle with sides $a = b = 1$ and $c = \sqrt{2}$ is right-angled,

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but $(1, 1, \sqrt{2})$ is not a Pythagorean triple because $\sqrt{2}$ is not an integer.

This article concerns integer triples of the form $(i, i + 1, k)$. Such triples were described in [1], and are also given in [2]. Closed expressions for such triples have been derived and described in the article in Mathworld (see [4]).

In the above mentioned works, a general formula for such triples was derived by solving the Pellian Diophantine. Our approach, however, will be more direct and will be based on first principles.

Another possible related description of such triples has been reported in [3].

2. Recurrence in such Triples

Let us consider the recurrence by which these triples are exhibited.

Now, we have to look for integer triads which satisfy

$$i^2 + (i+1)^2 = k^2 \,. \qquad (1)$$

A trivial example of such a triple is $(-1, 0, 1)$. Next, we also have $(3, 4, 5)$ and then $(20, 21, 29)$. For generating even more such triples, we can use a brute-force method. One such evaluation was carried out using Matlab, which yielded more such triples. The original program used is given in the Appendix. The output of the program is given in *Table* 1.

First, let us write (1) as below.

$$i^2 + (i+1)^2 = (i+d)^2 \ . \qquad (2)$$

It can be easily verified that

$$i = d - 1 \pm \sqrt{2d^2 - 2d} \,. \qquad (3)$$

Now denote $i = d - 1 + \sqrt{2d^2 - 2d}$ and $j = d - 1 - \sqrt{2d^2 - 2d}$. It is interesting to note that in (3), i will be

Table 1.

i	j	d	$i-j$
0	0	1	0
3	-1	2	4
20	-4	9	24
119	-21	50	140
696	-120	289	816
4059	-697	1682	4756
23660	-4060	9801	27720

an integer if d is of the form m^2 or m^2+1, m being any integer[1]. We use this fact to make our evaluation more efficient (see Appendix).

[1] This is because for $\sqrt{2d^2-2d}$ to be an integer, either d should be a square, and $2(d-1)$ should be a square, or, $d-1$ should be square and $2d$ should be square. From the above, d should be of the form m^2 or m^2+1.

Now, our brute force evaluation of such triples, and their corresponding d values are given in *Table* 1. Now consider *Table* 2, which gives us the recurrence in d. We list the values of m as a sequence:

$$1, 1, 3, 7, 17, 41, 99, ...$$

One notices that if $\{a_n\}_{n=0}^{\infty}$ denotes the above sequence, then the recurrence relation will be

$$a_{n+2} = 2a_{n+1} + a_n \,. \tag{4}$$

From this, we can find out a closed form expression for a_n in terms of an r. To do this, express each term in (4) in terms of an r, which is the unknown quantity. This is called '*ansatz*', i.e., an educated guess (see [5],[6]). That gives us

$$r^{n+2} = 2r^{n+1} + r^n \,. \tag{5}$$

Dividing by r^n on both sides, we get the quadratic:

$$r^2 - 2r - 1 = 0, \tag{6}$$

the solutions of which is

$$r = 1 \pm \sqrt{2}. \tag{7}$$

Table 2.

d		
1	=	1^2
2	=	1^2+1
9	=	3^2
50	=	7^2+1
289	=	17^2
1682	=	41^2+1
9801	=	99^2

Therefore,

$$r^n = (1 \pm \sqrt{2})^n.$$

Hence the closed-form expression for a_n is given by

$$a_n = C(1+\sqrt{2})^{n-1} + D(1-\sqrt{2})^{n-1}, \tag{8}$$

where C and D are undetermined constants. To find their values, we use $a_0 = a_1 = 1$ which we obtain from *Table* 2. After carrying out the necessary calculations and solving the system of equations, we get

$$C = \frac{1+\sqrt{2}}{2},\ D = \frac{1-\sqrt{2}}{2}.$$

On substituting these in (8), we get

$$a_n = \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2}. \tag{9}$$

We also note from *Table* 2 that,

$$d = \begin{cases} a_n^2 & \text{if } n \text{ is odd}\,, \\ a_n^2 + 1 & \text{if } n \text{ is even.} \end{cases}$$

Putting these values of d in (3), we get the following expression for i, j.

$$i = \begin{cases} a_n^2 \pm a_n\sqrt{2a_n^2 - 2} - 1 & \text{if } n \text{ is odd}\,, \\ a_n^2 \pm a_n\sqrt{2a_n^2 + 2} & \text{if } n \text{ is even}\,. \end{cases}$$

In all the above cases, $n \geq 0$.

3. Proof

To prove the above claims, we will show that the formulae for i and d satisfy (2). There are four different cases for i; let us first consider the case where n is odd, and positive sign is chosen before the radicand. Then we have

$$i^2+(i+1)^2 = 6a_n^4+4a_n^3\sqrt{2a_n^2-2}-6a_n^2-2a_n\sqrt{2a_n^2-2}+1. \tag{10}$$

All we need to prove is that $(i+d)^2$ is the same as (10). Putting $\beta = \sqrt{2a_n^2 - 2}$, we have

$$(i+d)^2 = ((a_n^2 + \beta a_n - 1) + a_n^2)^2$$

which is equal to

$$a_n^4 + a_n^2\beta^2 + 1 + 2a_n^3\beta - 2a_n^2 - 2a_n\beta + a_n^4 + 2a_n^2(a_n^2 + a_n\beta - 1),$$

that is,

$$6a_n^4 + 4a_n^3\beta - 6a_n^2 - 2a_n\beta + 1.$$

This is just the same as the right-hand side of equation (10). Similarly, we can prove the other three cases for i.

4. The Sequence 1,1,3,7... .

In this section, we consider some properties of the sequence for which a closed form formula was derived above. That sequence was

$$1, 1, 3, 7, 17, 41, 99, ...$$

Three interesting properties of the above sequence are given below, which establish the connection between this sequence, and the sequences for d, i, j and $i - j$.

- The difference of successive terms of the sequence for i form the odd-numbered terms of our sequence. The same is true for the sequence for j, the only difference being the change in the sign

- For the sequence for d, the difference of successive terms gives us the even numbered terms of our sequence.

- For the sequence for $i - j$, the difference of successive terms is the sum of odd numbered terms of our sequence.

We now consider the ratio of pairwise terms of our sequence, and observe the ratio,as $n \to \infty$. We consider the following.

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{a_{n+1}} &= \lim_{n\to\infty} \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}} \\ &= \lim_{n\to\infty} \frac{(1+\sqrt{2})^n}{(1+\sqrt{2})^{n+1}} \times \frac{1+(\frac{-1}{3+2\sqrt{2}})^n}{1+(\frac{-1}{3+2\sqrt{2}})^{n+1}} \\ &= \sqrt{2}-1 \ .\end{aligned}$$

5. Solutions of $i^2 + (i+1)^2 = (i+d)^2$

We show in this section that values of i and d obtained above are all the solutions to the equation $i^2+(i+1)^2 = k^2$. That is to say, we show that there are no values which satisfy the above equation and which are not determined by the above formulae for i and d. (Here $k = i + d$.)

Proof: We know that for the equation $x^2 + y^2 = z^2$, *all* the primitive integer solutions are given by

$$x = u^2 - v^2, \quad y = 2uv, \quad z = u^2 + v^2,$$

where u, v are integers. Now we can write $u^2 - v^2 = i$ and $u^2 + v^2 = i + d$. Using the values of i and d from above, and solving for u, v, we get the following. (Note that we consider here the case where n is even and positive sign is taken before the radicand. The other cases will have a similar proof.)

$$u = \sqrt{\frac{3a_n^2 + 2a_n\alpha + 1}{2}}, \quad v = \sqrt{\frac{a_n^2+1}{2}},$$

where $\alpha = \sqrt{2a_n^2 + 2}$. Now,

$$(2uv)^2 = 3a_n^4 + 2a_n^3\alpha + 4a_n^2 + 2a_n\alpha + 1.$$

Now this must be the same as $(i+1)^2$. So,

$$(i+1)^2 = (a_n^2 + a_n\alpha + 1)^2 = a_n^4 + a_n^2(2a_n^2 + 2) + 1 + 2a_n^3\alpha + 2a_n\alpha + 2a_n^2.$$

This is easily seen to be equal to

$$3a_n^4 + 2a_n^3\alpha + 4a_n^2 + 2a_n\alpha + 1$$

which was to be proved. So all the solutions to (2) are given by the formulae for i and d above.

Appendix

```
x=0;
 for m= 1:n
      d= m*m; i= (d-1) + sqrt(2*d*(d-1));
      j=(d-1)-sqrt(2*d*(d-1));
      if round(i)== i;round(j)==j;
          x=x+1;
          res(x,:)= [i,j,d,i-j];
      end
       for d= d+1;
      i= (d-1) + sqrt(2*d*(d-1));
      j=(d-1)-sqrt(2*d*(d-1));
       if round(i)== i;round(j)==j;
        x=x+1;
          res(x,:)= [i,j,d,i-j];
       end
       end
       end
       res=res
```

Here 'n' is any positive integer which is the upper limit for the evaluation.

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Suggested Reading

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