Symmetry Principles and Conservation Laws in Atomic and Subatomic Physics – 1

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The whole theoretical framework of physics rests only on a few but profound principles. Wigner enlightened us by elucidating that "It is now natural for us to try to derive the laws of nature and to test their validity by means of the laws of invariance, rather than to derive the laws of invariance from what we believe to be the laws of nature." Issues pertaining to symmetry, invariance principles and fundamental laws challenge the most gifted minds today. These topics require a deep and extensive understanding of both 'quantum mechanics' and the 'theory of relativity'. We attempt in this pedagogical article to present a heuristic understanding of these fascinating relationships based only on rather elementary considerations in classical and quantum mechanics. An introduction to some fundamental considerations regarding continuous symmetries, dynamical symmetries (Part 1), and discrete symmetries (Part 2) (parity, charge conjugation and time-reversal), and their applications in atomic, nuclear and particle physics, will be presented.


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1. Introduction

The principal inquiry in classical mechanics is to seek a relationship between position, velocity, and acceleration. This relationship is rigorously expressed in what we call the 'equation of motion'. The equation of motion is not self-evident, but rests on some fundamental principle that must be discovered. A prerequisite for this discovery is the principle of inertia, discovered by

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Galileo, contrary to common experience, that the velocity of an object is self-sustaining and remains invariant in the absence of its interaction with an external agency. This principle identifies an inertial frame of reference in which physical laws apply. This great discovery by Galileo was soon incorporated in Newton's scheme as the First law of mechanics, the law of inertia. Newton recognised, following his invention of calculus, that it is the *change* in velocity that seeks a cause. Newton's calculus expressed the rate of change of velocity as acceleration which is interpreted as the 'effect' of the physical interaction that generated it. Newton's second law expresses this 'cause-effect' relationship as a linear response of the system to the physical interaction it experienced: $\overrightarrow{F} = m\overrightarrow{a}$. The mass m of the object is the constant of proportionality between the effect ($\overrightarrow{a}$) and the cause ($\overrightarrow{F}$).

[1]This article is partly based on the talk given by PCD at the Karnataka Science and Technology Academy's special lectures at the Bangalore University on 23rd March, 2009.

In the following section we will begin by considering how Newton's third law introduces a simple illustration of the relation between a symmetry and a conservation law. In the remainder of the article we will explore similar relationships that impact much of the frontiers of physics, which are being investigated today; these studies use powerful theoretical frameworks and sophisticated technology.

2. Translational Invariance and Conservation of Momentum

We consider a closed system of N point particles in homogeneous isotropic space. The force on the kth particle is the sum of forces on it due to all the other particles

Newton's third law introduces a simple illustration of the relation between a symmetry and a conservation law.

$$\overrightarrow{F}_k = \sum_{\substack{j=1 \\ j \neq k}}^{N} \overrightarrow{f}_{kj} \, . \tag{1}$$

We now consider 'virtual' translational displacement of the entire N-particle system in the homogeneous space.

In such a process, the internal forces can do no work, since the process amounts to merely displacing the entire system to an adjacent region, displaced from the original by an amount $\overrightarrow{\delta s}$. As this displacement is being considered in a homogeneous medium, it is referred to as being 'virtual' as no work is done by the internal forces. This phenomenon is then expressed by the relation

$$\sum_{k=1}^{N} \overrightarrow{F}_k \cdot \overrightarrow{\delta s} = \sum_{k=1}^{N} \frac{\mathrm{d}\overrightarrow{P_k}}{\mathrm{d}t} \cdot \overrightarrow{\delta s} = 0 \,, \tag{2}$$

where $\overrightarrow{P_k}$ is the momentum of the kth particle. In expressing this quantitative result, we have made use of Newton's first two laws (the first law implicitly and the second law explicitly) and also the notion of translational invariance in homogeneous space. Now, for an arbitrary displacement $\overrightarrow{\delta s}$, this relationship requires that

$$\sum_{k=1}^{N} \frac{\mathrm{d}\overrightarrow{P_k}}{\mathrm{d}t} = 0 \,. \tag{3}$$

If we write this result for a two-body closed system, we discover Newton's third law, that action and reaction are equal and opposite:

$$\frac{\mathrm{d}\overrightarrow{P_1}}{\mathrm{d}t} = -\frac{\mathrm{d}\overrightarrow{P_2}}{\mathrm{d}t} \,. \tag{4}$$

In other words, we discover that conservation of linear momentum is governed by the symmetry principle of translational invariance in homogeneous space. Likewise, one can see that the conservation of angular momentum emerges from rotational displacements in isotropic space.

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It is interesting to observe that Newton actually invented calculus to explain departure from equilibrium of an object which manifests as its acceleration, and proposed a linear relationship between the physical interaction (force) which he interpreted as the 'cause' of the acceleration. Newton's second law contains the heart of this stimulus–response relation, expressed as a differential equation. It is interesting that laws of classical mechanics can be built alternatively on the basis of an 'integral principle', namely the 'principle of variation', discussed in the next section.

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3. Principle of Variation

The connection between symmetry and conservation laws becomes even more transparent in the alternative formalism of classical mechanics, namely the Lagrangian/ Hamiltonian formulation. It is instructive to first see that this alternative formalism is based *not* on the linear response relationship embodied in the Newtonian principle of causality, but in a completely different approach, namely the 'principle of variation'.

Newtonian mechanics offers an accurate description of classical motion by accounting for the same by the 'cause and effect' relationship. An alternative and equivalent description makes it redundant to invoke such a causal description. This alternative description dispenses the Newtonian notion of the 'cause-effect' relationship, and instead of it invokes a variational principle, namely, that the 'action integral' is an extremum. Those who are used to thinking in terms of the Newtonian formulation alone would find it strange that one gets equivalent description of classical mechanics without invoking the notion of force at all!

Let us first state the principle of extremum action. We begin on common ground with the Newtonian formulation, namely that the position q and velocity $\dot{q}$ specify the mechanical state of a system. A well-defined function of q and $\dot{q}$ would also then specify the mechanical state of the system. What is known as the Lagrangian of a system $\mathcal{L}(q, \dot{q})$ is just that; it is named after its originator Lagrange (1736–1813). Furthermore, in a homoge-

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$$S = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)\mathrm{d}t \ ,$$

is an extremum.

neous isotropic system, $\mathcal{L}(q, \dot{q})$ can depend only quadratically on the velocity, so that it could be independent of its direction. The simplest form the Lagrangian would then have is $\mathcal{L}(q, \dot{q}) = f_1(q) + f_2(\dot{q}^2)$, wherein the functions f_1 and f_2 must be suitably chosen. It turns out that the choice $f_1(q) = -V(q)$, i.e., the negative of the particle's potential energy, and $f_2(\dot{q}^2) = (m/2)\dot{q}^2$, i.e., the kinetic energy T of the particle, renders this new formalism completely equivalent to Newtonian mechanics. This relationship offers us with a simple interpretation of the Lagrangian as $\mathcal{L}(q, \dot{q}) = T - V$.

Simply stated, the principle of least action is that the mechanical state of a system evolves along a world-line[2] such that the 'action',

$$S = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)\mathrm{d}t \ , \qquad (5)$$

[2] A 'world-line' is a trajectory in the phase space, or the mathematical space, along which the mechanical state of a system evolves over a period of time.

is an extremum. This principle was formulated by Hamilton (1805–1865). It has an interesting development beginning with Fermat's principle about how light travels between two points, and subsequent contributions by Maupertius (1698–1759), Euler (1707–1783), and Lagrange himself. The principle that 'action' is an extremum is equivalent to stating that the mechanical system evolves over the period t_1 to t_2 along a world-line traced by the points $(q, \dot{q})$ such that if the 'action integral' S is evaluated along any other alternative path displaced infinitesimally from the one it actually evolves over, $(q + \delta q, \dot{q} + \delta\dot{q})$, then:

$$\delta S = \int_{t_1}^{t_2} \mathcal{L}(q + \delta q, \dot{q} + \delta\dot{q}, t)\mathrm{d}t - \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)\mathrm{d}t = 0. \qquad (6)$$

The above equation is a mathematical expression of the statement of the 'principle of extremum action'. The necessary and sufficient condition that this principle must hold good provides us the well-known Lagrange's equation of motion:

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}} = 0. \qquad (7)$$

The quantity $(\partial\mathcal{L})/(\partial\dot{q})$ in the above equation is known as the *generalised momentum* (written as p) conjugate to the generalised coordinate q. The power of Lagrangian mechanics comes from the fact that there are very many pairs of variables (q, p) which can be considered conjugate to each other in the Lagrangian sense – q and p need not have the physical dimensions of [L] and [MLT^{-1}] respectively. The dimension of the product of q and p, however, must always be $\mathrm{ML}^2\mathrm{T}^{-1}$, that of the angular momentum. From Lagrange's equation, it follows immediately that if the Lagrangian is independent of q, (i.e., if $(\partial\mathcal{L})/(\partial q) = 0$) then the generalized momentum $p = (\partial\mathcal{L})/(\partial\dot{q})$ conjugate to this coordinate is constant. The independence of the Lagrangian with respect to q is an expression of 'symmetry', since the Lagrangian would then be the same no matter what the value of q is. This results in a conservation principle since the generalised momentum conjugate to this q becomes independent of time, remains constant. One may pair (time, energy) as (q, p), and see from this that $(\partial\mathcal{L}/(\partial t) = 0$ would result in energy being constant. This result immediately follows from the following expression for the time-derivative of the Lagrangian:

The independence of the Lagrangian with respect to q is an expression of 'symmetry', since the Lagrangian would then be the same no matter what the value of q is. This results in a conservation principle since the generalised momentum conjugate to this q becomes independent of time, remains constant.

$$\begin{aligned} 0 = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} &= \frac{\partial\mathcal{L}}{\partial q}\dot{q} + \frac{\partial\mathcal{L}}{\partial\dot{q}}\ddot{q} + \frac{\partial L}{\partial t} \\ &= \left\{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}}\right\}\dot{q} + \frac{\partial\mathcal{L}}{\partial\dot{q}}\ddot{q} + \frac{\partial L}{\partial t}\ , \end{aligned} \qquad (8)$$

where Lagrange's equation is used to re-express the first term.

It thus follows that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\mathcal{L}}{\partial\dot{q}}\dot{q} - \mathcal{L}\right] = -\frac{\partial L}{\partial t}\ . \qquad (9)$$

Conservation of energy follows from the symmetry principle that the Lagrangian is invariant with respect to time.

From the above, it immediately follows that when the Lagrangian depends on time only implicitly through its dependence on q and $\dot{q}$, then:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{q} - \mathcal{L}\right] = 0 \, , \tag{10}$$

which implies $\left[\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\dot{q} - \mathcal{L}\right]$ is a conserved quantity. This quantity is called the Hamiltonian, or Hamilton's principal function, of the system, which for a conservative system is essentially the same as the total energy of the system. This can be seen easily by identifying the generalized momentum and substituting $T - V$ for the Lagrangian. We thus see that conservation of energy follows from the symmetry principle that the Lagrangian is invariant with respect to time.

These results illustrate an extremely powerful theorem in physics, known as the Noether's theorem, which can be stated informally as:

If a system has a continuous symmetry property, then there are corresponding quantities whose values are conserved in time [1].

This theorem is named after Noether (1882–1935), of whom Einstein said:

In the judgement of the most competent living mathematicians, Fräulein Noether was the most significant creative mathematical genius thus far produced since the higher education of women began [2].

If a system has a continuous symmetry property, then there are corresponding quantities whose values are conserved in time.

4. Symmetry Principles and Physical Laws

We have now seen that both the equation of motion and the conservation principles result from the *single* principle of least action. Moreover, the same principle provides for the connection between symmetry and conservation laws, exalted by Noether to one of the most profound principles in contemporary physics. We now ask if the conservation principles are consequences of the laws of Nature, or, rather the laws of Nature are consequences of the symmetry principles that govern them?

Until Einstein's special theory of relativity, it was believed that conservation principles are the result of the laws of Nature. Since Einstein's work, however, physicists began to analyze the conservation principles as consequences of certain underlying symmetry considerations from which they could be deduced, enabling the laws of Nature to be revealed from this analysis. Wigner's profound impact on physics is that his explanations of symmetry considerations using 'group theory' resulted in a change in the very perception of just what is most fundamental, and physicists began to regard 'symmetry' as the most fundamental entity whose form would govern the physical laws. Wigner was awarded the 1963 Nobel Prize in Physics for these insights [3].

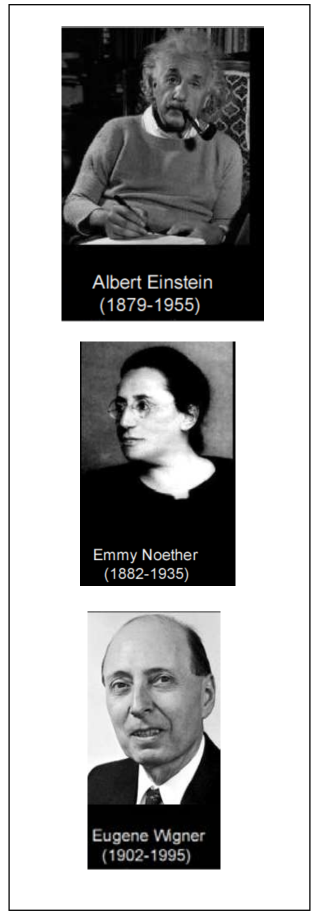

Figure 1. Masters of symmetry.

The conservation of linear and angular momentum we illustrated above are consequences of invariance under continuous displacements and rotations respectively in homogenous and isotropic space. Likewise, the conservation of energy is a consequence of invariance under continuous temporal displacement.

A detailed exposition of the governing symmetry principles requires group theoretical methods, and is clearly beyond the scope of this article, but we continue to dwell on some other kinds of symmetries now and examine their connections with conservation principles.

5. Dynamical Symmetry: Laplace–Runge–Lenz Vector

It is well known that in the classical two-body Kepler problem (gravitational Sun–Earth system, or the Coulombic proton–electron planetary model of the old-quantum-theory of the hydrogen atom), both energy and angular momentum are conserved. We have already discussed

Refer to *Resonance* issues on:
Einstein, Vol.5, March and April 2000.
Noether, Vol.3,September 1998.
Wigner, Vol.14, October 2009.

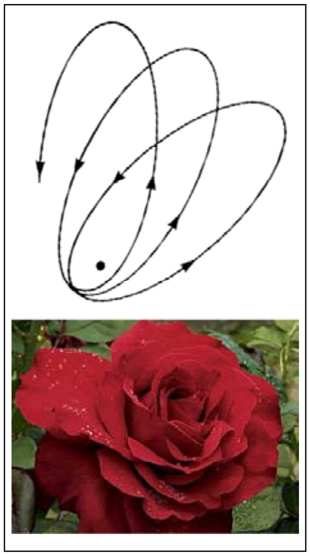

Figure 2. If the eclipse were to precess it would generate what is called a 'rosette' motion since the trajectory of the planet would seem to go over the petals of a rose, if seen from a distance.

the associated symmetries. What is interesting is that the elliptic orbit of the Kepler system for bound states is fixed, i.e., the ellipse does not precess (*Figure* 2).

Can we then find a symmetry that would explain the constancy of the orbit? It turns out that the orbit itself remains fixed if and only if the potential in which motion occurs is strictly of the form $-1/r$ and the associated force is of the form $-1/r^2$. This is true for both the gravitational and the Coulomb potential, and hence the Kepler elliptic orbits remain fixed. This is rigorously expressed as the constancy of the Laplace–Runge–Lenz (LRL) vector. The LRL vector is defined as:

$$\vec{A} = \left(\vec{v} \times \vec{H}\right) - \kappa\, \hat{e}_\rho \qquad (11)$$

and is shown in *Figure* 3 [4]. In the above equation $\vec{v}$ is the 'specific' linear momentum and $\vec{H}$ is the 'specific' angular momentum. The term 'specific' denotes the fact that the physical quantities linear momentum and angular momentum, which are being referred to, are defined per unit mass. Likewise in the second term of the LRL vector, κ is the proportionality in the inverse distance gravitational potential per unit mass of the planet. It can be easily verified that the time derivative of the LRL vector vanishes, and the $\vec{A}$ is therefore a conserved quantity. Its direction is from the focus of the ellipse to the perihelion (*Figure* 3) [4], which has a direction along the major axis of the ellipse, thus holding the ellipse fixed.

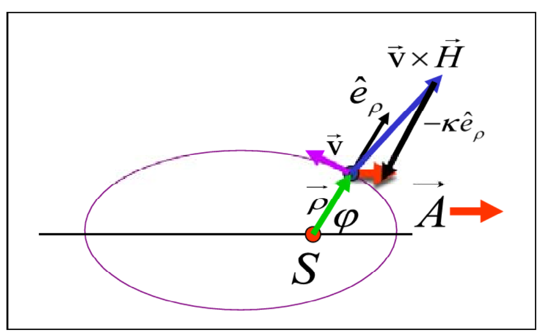

Figure 3. Schematic diagram showing the Laplace–Runge–Lenz vector, $\vec{A}$.

The constancy of the LRL vector is a conservation principle, and since the governing criterion involves dynamics (namely that the force must have a strict inverse square form), the associated symmetry is called 'dynamical symmetry'. Sometimes, it is also called an 'accidental' symmetry. This symmetry breaks down when there is even a minor departure from the inverse square law force, as would happen in a many-electron atom, such as the hydrogen-like sodium atom. The potential experienced by the 'outer-most' electron goes as $1/r$ only in the asymptotic ($r \to \infty$) region. Close to the center, the potential goes rather as $-Z/r$, due to the reduced screening of the nuclear charge by the orbital electrons, and thus departs from $1/r$. This difference in the hydrogen atom potential and that in the sodium atom is due to the quantum analogue of the breakdown of the LRL vector constancy in the sodium atom. Using group theoretical methods, Vladmir Fock (1898–1974) explained the dynamical symmetry of the hydrogen atom [5].

Using the language of group theory, the Fock symmetry accounts for the $(2l + 1)$-fold degeneracy of the hydrogen atom eigenstates. This degeneracy is lifted for the hydrogen-like sodium atom due to the breakdown of the associated symmetry. In atomic physics, this is often expressed in terms of what is called as 'quantum defect' $\mu_{n,l}$ which makes the hydrogenic energy eigenvalues depend not merely on the principal quantum number n but also on the orbital angular momentum quantum number l. This enables the use of the hydrogenic formula for energy with n replaced by $n_{\text{effective}} = n - \mu_{n,l}$. The 'quantum-defect theory' has very many applications in the analysis of the atomic spectrum, including the 'autoionization resonances' [6,7]. As pointed out above, the conservation of angular momentum is due to the rotational symmetry, referred to as the symmetry under the group SO(3). All central fields have this symmetry. However, the inverse-square-law force (such as gravity or

The conservation of the generalized momentum which is conjugate to a cyclic coordinate is a generic expression of a deeper relationship between symmetry and conservation laws.

Coulomb) has symmetry under a bigger group, SO(4) or SO(3; 1), where SO(4) is the rotational group in 4 dimensions, and SO(3; 1) is the Lorentz group. The dimensionality of the SO(N) group is $N(N-1)/2$, so the SO(4) group is 6-dimensional and corresponds to the 6 conserved quantities, namely the 3 components of the angular momentum vector and the three components of Pauli–Runge–Lenz vector which is the quantum analogue of the LRL vector [8].

6. Conclusion

The conservation of the generalized momentum which is conjugate to a cyclic coordinate is a generic expression of a deeper relationship between symmetry and conservation laws. In the next part of this article we shall discuss discrete symmetries, the CPT symmetry and comment on spontaneous symmetry breaking and the search for the Higgs boson.

Suggested Reading

[1] W J Thompson, *Angular Momentum*, Wiley, p.5, 2004.
[2] From a letter to the *New York Times* on May 5th, 1935 from Albert Einstein shortly after Emmy Noether's death.
[3] Details of the 1963 Nobel Prize in physics can be found at http://nobelprize.org/nobel_prizes/physics/laureates/1963/index.html
[4] For a detailed discussion of the Laplace–Runge–Lenz vector see H Goldstein, *Classic Mechanics*, Second Edition, Addison-Wesley, p102ff, 1980.
[5] W Fock, *Z. Phys.*, Vol.98, p.145, 1935.
[6] M J Seaton, *Rep. Prog. Phys.*, Vol.46, p.167, 1983.
[7] S B Whitfield, R Wehlitz, H R Varma, T Banerjee, P C Deshmukh and S T Manson, *J. Phys. B: At. Mol. Opt. Phys.*, Vol.39, p.L335, 2006.
[8] V Bargmann, *Z. Physik* Vol.99, pp.576–582, 1936.

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