This and a companion paper by Ferdinand Freudenstein, published in mid 1950s when graphical methods were prominent, defined the direction of modern kinematics research by introducing algebraic methods. The systematic methods presented in this paper were sufficiently general to enable others extend them to numerous mechanism synthesis problems in the half a century that followed. This paper is a required reading for a kinematician. It is conceptually rich and pays due attention to practical aspects.

G K Ananthasuresh

Approximate Synthesis of Four-Bar Linkages *[1,2]

By Ferdinand Freudenstein **[3],** ***New York, N.Y.***

Formulas are presented for obtaining the characteristics of a four-bar linkage, designed to generate an arbitrary function approximately over a finite range. A number of methods of varying degrees of accuracy and complexity have been developed, enabling a designer to select the one best suited to his requirements.

Nomenclature

The following nomenclature is used in the paper (see Fig. 1); $ABCD$ represents a 4-bar linkage in which crank $AB = b$, crank $CD = d$, connecting rod $BC = c$, and fixed link $DA = 1$. All angles are measured in radians unless otherwise stated. Length units are arbitrary.

ϕ = $\angle XAB$, measured clockwise from AX to AB
ψ = $\angle ADC$, measured clockwise from DA to DC

* Reprinted with permission from *Transactions of the ASME*, Vol.77, Aug. 1955, pp.853–861.

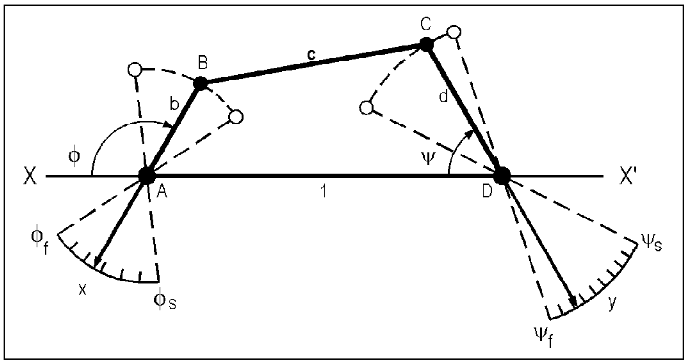

Fig. 1. Four-Bar Linkage.

$y = f(x)$	$=$	function to be generated, ideal function $(x_s \leq x \leq x_f)$
$y = g(x)$	$=$	function actually generated, actual function
x_s, y_s	$=$	starting values of ideal function, corresponding to ϕ_s, ψ_s
x_f, y_f	$=$	final values of ideal function, corresponding to ϕ_f, ψ_f
$y_f - y_s$	$=$	$\Delta y =$ range of y, corresponding to $\psi_f - \psi_s = \Delta\psi$
$x_f - x_s$	$=$	Δx, range of x, corresponding to $\phi_f - \phi_s = \Delta\phi$
ϕ_0, ψ_0	$=$	values of ϕ, ψ corresponding to $x = y = 0$
$r_\phi = \Delta\phi/\Delta x$	$=$	input-scale factor
$r_\psi = \Delta\psi/\Delta y$	$=$	output-scale factor
R_i	$=$	ith bar ratio of 4-bar linkage
e	$=$	maximum inherent error $= (\lvert f(x) - g(x)\rvert_{\max})/\Delta y(\max)$
m_i	$=$	$d^i\psi/d\phi^i = i$th derivative of ψ with respect to ϕ
x_m, y_m	$=$	values of x, y at precision point with precision derivatives, corresponding to ϕ_m, ψ_m

Introduction

In the design of computing mechanisms it is often desired to generate arbitrary functions, in which the variables are represented by an analogous quantity such as a shaft rotation. Gear and cam mechanisms have been employed for this purpose. In recent times, consideration has been given to the use of bar linkages for function generation, in particular in connection with the development of fire-control devices.

Linkages are inherently light, inexpensive, adaptable to high speeds, and have little friction. The class of functions suitable for linkage representation is large. Mechanisms involving a finite number of links possess an inherent error and it is the task of the designer to reduce this error to a sufficiently low value. A successful design procedure must combine the predominantly analytical considerations of linkage synthesis with practical considerations involving mechanical advantages, ranges, dead-center positions, friction, and backlash.

The large number of variables occurring in the synthesis of even a single 4-bar linkage and the complexity of their interaction, render an analytical treatment difficult. The development of that portion of the field of linkage synthesis pertaining to function generation therefore has been primarily graphical [4,5,6]. This paper will present analytical methods. Some of these can be combined with the powerful techniques developed by Svoboda [4], and it is hoped that the material presented will lead toward the simplification and systematization of the synthesis process.

Variables Involved in Mechanization of an Arbitrary Function

Let it be required to generate the function $y = f(x)$ in the interval $x_s \leq x \leq x_f$, $y_s \leq y \leq y_f$. In the case of a single 4-bar linkage seven variables determine the generated function; three independent bar ratios, R_1, R_2, R_3, which define the proportions of the linkage; two scale factors, r_ϕ, r_ψ; and two zero values ϕ_0, ψ_0.

The relation which the linkage is designed to generate is

$$y = \frac{\psi - \psi_0}{r_\psi} = f(x) = f\left(\frac{\phi - \phi_0}{r_\phi}\right). \qquad (1)$$

If the values of some variables are determined prior to the synthesis, the remaining ones should then be so chosen as to effect the best approximation to the ideal function 1.

Choice of Arbitrary Variables

Intelligent choice of arbitrary variables is required to obtain linkages having reasonable proportions and ranges. As a rough rule, the ratio of the crank lengths is inversely proportional to the ratio of the ranges. In general, ranges up to about 120 deg are feasible. These determine the scale factors, which in turn determine the mechanical advantage of the ideal linkage. Harmonic transformer-type linkages are characterized by a linear relationship between $\cos\phi$ and $\cos\psi$. The existence of such a relationship can be determined graphically. It will be clear from Equation (2) that such relations are not obeyed by 4-bar linkages and should be avoided.

Basic Considerations

In synthesis investigations, a means of understanding linkage behaviour may be expected to be furnished by the shortest relation between ϕ and ψ involving the minimum number of side ratios. Using the vector equation $BC^2 = (AB + CD + DA) \cdot (AB + CD + DA)$ and noting that the angle between AB and CD is $(\phi - \psi)$, one obtains for $a = 1$

$$R_1 \cos\phi - R_2 \cos\psi + R_3 = \cos(\phi - \psi) \qquad (2)$$

where

$$R_1 = 1/d \qquad (3)$$

$$R_2 = 1/b \qquad (4)$$

$$R_3 = (1 + b^2 - c^2 + d^2)/(2bd) \qquad (5)$$

R_1, R_2 and R_3 are three independent bar ratios. Relation (2) is believed to have the desired simplicity.

Two types of approximations will be developed from Equation (2). In the first, the ideal function and the actual function are made to coincide at several points, termed precision points. Between precision points, the actual function will differ from the ideal function by an amount depending upon the distance between precision points and upon the nature of the ideal function. The "3-point approximation" is a rapidly found approximation with three precision points. For greater accuracy, more elaborate 4 and 5-point approximations have been developed.

In the second type of approximation, the ideal and actual functions coincide at only one precision point, at which, however, a number of derivatives of the actual function also will coincide with those of the ideal function.

Method of Determining Desired Approximations

The ideal values of x, y, and their derivatives at the precision points are substituted into Equation (2) or into its differentiated form Equation (4.1), Table 4. Several simultaneous equations in the unknowns $R_{1,2,3}, \phi_s$, and ψ_s

are thus obtained. Since the independent bar ratios occur only once each in Relation (2), the linkage defined by the simultaneous equations will assume the ideal values employed in the substitutions.

The 3-Point Approximation (Table 1)

Let it be required to design a linkage generating $y = f(x)$ with three precision points corresponding to the pairs of angles (ϕ_i, ψ_i) $i = 1, 2, 3$. On substituting the values of these angles in Relation (2), three linear simultaneous equations are obtained in the unknowns $R_{1,2,3}$. The solutions are shown in Table 1. The lengths of sides b, c, d are determined from the R's

Table 1. Three-Point Approximation.

Linkage to pass through $(\phi_1, \psi_1), (\phi_2, \psi_2)$, and (ϕ_3, ψ_3)

$$R_1 \cos\,\phi_i - R_2 \cos\psi_i + R_3 = \cos\,(\phi_i - \psi_i), \quad i = 1, 2, 3\ . \tag{1.1}$$

Let

$$\cos\phi_1 - \cos\phi_2 = \omega_1 \tag{1.2}$$

$$\cos\phi_1 - \cos\phi_3 = \omega_2 \tag{1.3}$$

$$\cos\psi_1 - \cos\psi_2 = \omega_3 \tag{1.4}$$

$$\cos\psi_1 - \cos\psi_3 = \omega_4 \tag{1.5}$$

$$\cos(\phi_1 - \psi_1) - \cos(\phi_2 - \psi_2) = \omega_5 \tag{1.6}$$

$$\cos(\phi_1 - \psi_1) - \cos(\phi_3 - \psi_3) = \omega_6 \tag{1.7}$$

$$R_1 = \frac{\omega_3\omega_6 - \omega_4\omega_5}{\omega_2\omega_3 - \omega_1\omega_4} \tag{1.8}$$

$$R_2 = \frac{\omega_1\omega_6 - \omega_2\omega_5}{\omega_2\omega_3 - \omega_1\omega_4} \tag{1.9}$$

$$R_3 = \cos(\phi_i - \psi_i) + R_2 \cos\psi_i - R_1 \cos\phi_i \tag{1.10}$$

Error in ψ for a given value of ϕ,

$$\approx \frac{1}{6}(\phi - \phi_1)(\phi - \phi_2)(\phi - \phi_3)\left[\frac{r\psi}{r\phi^3}y''' - \frac{d^3\psi}{d\phi^3}\right] \tag{1.11}$$

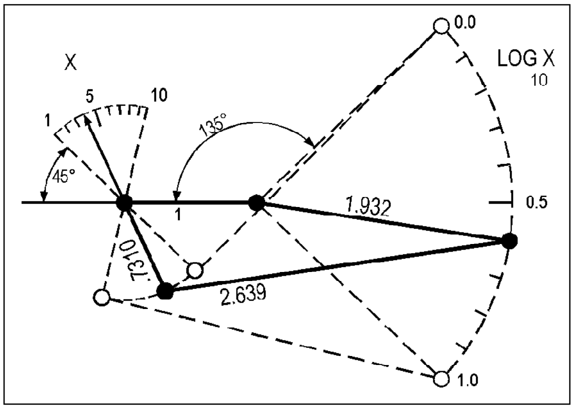

Fig. 2. Three-Point Approximation for Log Function.

using Relations (3,4,5). A qualitative estimate of the error can be derived from Expression (1.11), which is based on the assumption that the error in ψ can be represented by the expression $L(\phi-\phi_1)\times(\phi-\phi_2)(\phi-\phi_3)$, where L is so chosen that the error vanishes in four places, which fact serves to evaluate L. Similar expressions are derivable for the error in approximations having a larger number of precision points.

Example. $y = f(x) = \log_{10} x$, $1 \le x \le 10$, with precision points at $x = 1, 3, 10$. Suppose $\Delta\phi = 60°$, $\Delta\psi = 90°$, $\phi_s = 45°$, $\psi_s = -45°$. With the aid of Table 1, $a = 1$, $b = -0.7310$, $c = 2.6391$, $d = -1.9319$; $e = 6$ per cent at $x = 1.35$. The linkage is shown in Fig. 2. The negative values of the cranks b, d are to be interpreted in the vector sense. The spacing of the precision points for a minimum e in an n-point approximation is believed to be such that the identical e is reached at least once between each pair of precision points and between end points and the nearest precision points.

The 4-Point Approximation (Table 2)

In this case the scale factors and the initial value of $(\phi - \psi)$ have been considered arbitrary. The quantities $R_{1,2,3}$ and ϕ_s are to be determined with the aid of Equation (2), which assumes the form Equations (2.3) (Table 2) at each of the four precision points.

Subtracting the first Equations (2.3) from the remaining to eliminate R_3 and using the identity.

$$\cos x - \cos y = -2\sin 1/2(x+y)\sin 1/2(x-y)$$

Table 2. Four-Point Approximation.

$y = f(x)$ to be generated with precision points at (x_i, y_i), $i = 1, 2, 3, 4$. Choose values for $r_\phi, \phi_s - \psi_s$

Let

$$p_i = r_\phi(x_i - x_s), \quad i = 1, 2, 3, 4 \tag{2.1}$$

$$q_i = r_\psi(y_i - y_s), \quad i = 1, 2, 3, 4 \tag{2.2}$$

$$R_1 \cos(\phi_s + p_i) - R_2 \cos(\psi_s + q_i) + R_3 = \cos[(\phi_s + p_i) - (\psi_s + q_i)] \tag{2.3}$$

Let

$$\alpha_i = \frac{1}{2}(p_1 + p_i), \quad i = 1, 2, 3, 4, \quad \text{from now on} \tag{2.4}$$

$$\beta_i = \frac{1}{2}(q_1 + q_i) \tag{2.5}$$

$$A_i = \sin(\alpha_i - p_1) \tag{2.6}$$

$$B_i = \sin(\beta_i - q_1) \tag{2.7}$$

$$C_i = \sin[(\alpha_i - p_1) - (\beta_i - q_1)] \tag{2.8}$$

$$\delta_i = \alpha + \alpha_i \tag{2.9}$$

$$\alpha = \phi_s - \psi_s \tag{2.10}$$

$$\lambda_i = C_i \sin(\delta_i - \beta_i) \tag{2.11}$$

Then Equation (2.3) becomes

$$R_1 A_i \sin(\psi_s + \delta_i) - R_2 B_i \sin(\psi_s + \beta_i) = \lambda_i \tag{2.12}$$

where $i = 2, 3, 4$ and unknowns are R_1, R_2 and ψ_s. Let

$$k_1 = B_3\lambda_2 \cos\beta_3 - B_2\lambda_3 \cos\beta_2 \tag{2.13}$$

$$k_2 = B_3\lambda_2 \cos\beta_3 - B_2\lambda_3 \sin\beta_2 \tag{2.14}$$

$$l_i(4, 3) = k_i(3, 2) \text{ (nos. refer to subscripts)} \tag{2.15}$$

$$r_1 = B_3 A_2 \cos\delta_2 \cos\beta_3 - B_2 A_3 \cos\delta_3 \cos\beta_2 \tag{2.16}$$

$$r_2 = B_3 A_2 \sin\delta_2 \sin\beta_3 - B_2 A_3 \sin\delta_3 \sin\beta_2 \tag{2.17}$$

$$r_3 = B_3 A_2 \sin(\delta_2 + \beta_3) - B_2 A_3 \sin(\delta_3 + \beta_2) \tag{2.18}$$

Table 2 continued...

$$s_i(4,3) = r_i(3,2) \quad \text{(nos. refer to subscripts)} \tag{2.19}$$

$$m_1 = k_1 s_1 - l_1 r_1 \tag{2.20}$$

$$m_2 = k_2 s_1 + k_1 s_3 - l_2 r_1 - l_1 r_3 \tag{2.21}$$

$$m_3 = k_1 s_2 + k_2 s_3 - l_1 r_2 - l_2 r_3 \tag{2.22}$$

$$m_4 = k_2 s_2 - l_2 r_2 \tag{2.23}$$

$$m_1 \tan^3 \psi_3 + m_2 \tan^2 \psi_s + m_3 \tan \psi_s + m_4 = 0 \tag{2.24}$$

$\psi_s = -\beta_3$ is a trivial (meaningless) solution of Equation (2.24) and can be used to reduce Equation (2.24) to a quadratic equation in $\tan \psi_s$.

Equations (2.12) are obtained, in which the unknowns are $R_{1,2}$ and ψ_s. By eliminating $R_{1,2}$ from Equations (2.12), a cubic equation in $\tan \psi_s$ (2.24), is eventually obtained. Owing to the nature of the elimination process, $\psi_s = -\beta_3$ is a meaningless solution, which can be used to reduce Equation (2.24) to a quadratic form. Knowing ϕ_s and ψ_s, the R's are found as in the 3-point approximation.

Example. $y = \log_{10} x$, as in the preceding example and $x = 1, 4, 7, 10$ are precision points. Suppose $\alpha = 60°$, $r_\phi = 7.5°/x$, $r_\psi = 75°/y$. Then $a = 1$, $b = -1.599$, $c = 2.841$, $d = 2.442$, $\phi_0 = -11°13'$, $\psi_0 = -63°43'$. $e = 3.6$ per cent at $x = 2$. The linkage is shown in Fig. 3.

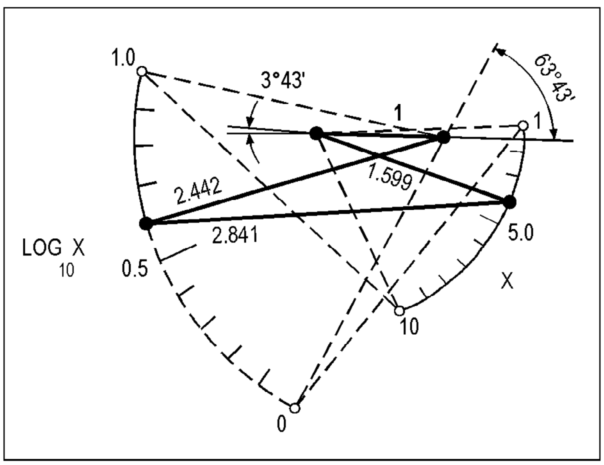

Fig. 3. Four-Point Approximation for Log Function.

The 5-Point Approximation (Table 3)

The arbitrary variables in this approximation are the scale factors. Up to Equation (2.8), the procedure is the same as in the 4-point approximation. Eliminating R_2 between the first and second and also between the third and fourth of Equations (3.1) and R_1 from the resulting two expressions, one obtains eventually an equation relating $\tan\phi_s$ and $\tan\psi_s$, Equation (3.31). Similarly, eliminating first R_1 and then R_2, another relation between $\tan\phi_s$ and $\tan\psi_s$ is derivable, Equation (3.32). These are higher-order simultaneous equations and can be solved graphically. Knowing ϕ_s and ψ_s, the solution can be completed as in a 3-point approximation. It is advisable to use at least 7-figure accuracy in such an evaluation.

Table 3. Five-Point Approximation.

$y = f(x)$ to be generated with precision points at (x_i, y_i), $i = 1, 2, 3, 4$. Choose values for r_ϕ and r_ψ.

Equations (2.1) through (2.8) of Table 2 remain unchanged, but the maximum value of i is now 5.

$$R_1 A_i \sin(\phi_s + \alpha_i) - R_2 B_i \sin(\psi_s + \beta_i) = C_i \sin[(\phi_s + \alpha_i) - (\psi_s + \beta_i)] \tag{3.1}$$

where $i = 2, 3, 4, 5$ from now on. Let

$$P_1 = C_2 B_3 \cos\beta_3 \cos(\alpha_2 - \beta_2) - C_3 B_2 \cos\beta_2 \cos(\alpha_3 - \beta_3) \tag{3.2}$$

$$P_2 = C_2 B_3 \cos\beta_3 \sin(\alpha_2 - \beta_2) - C_3 B_2 \cos\beta_2 \sin(\alpha_3 - \beta_3) \tag{3.3}$$

$$P_3 = C_2 B_3 \sin\beta_3 \cos(\alpha_2 - \beta_2) - C_3 B_2 \sin\beta_2 \cos(\alpha_3 - \beta_3) \tag{3.4}$$

$$P_4 = C_2 B_3 \sin\beta_3 \sin(\alpha_2 - \beta_2) - C_3 B_2 \sin\beta_2 \sin(\alpha_3 - \beta_3) \tag{3.5}$$

$$Q_i(5,4) = P_i(3,2) \quad \text{(nos. refer to subscripts)} \tag{3.6}$$

$$S_1 = A_2 B_3 \cos\alpha_2 \cos\beta_3 - A_3 B_2 \cos\alpha_3 \cos\beta_2 \tag{3.7}$$

$$S_2 = A_2 B_3 \cos\alpha_2 \sin\beta_3 - A_3 B_2 \cos\alpha_3 \sin\beta_2 \tag{3.8}$$

$$S_3 = A_2 B_3 \sin\alpha_2 \cos\beta_3 - A_3 B_2 \sin\alpha_3 \cos\beta_2 \tag{3.9}$$

Table 3. continued...

$$S_4 = A_2 B_3 \sin\alpha_2 \sin\beta_3 - A_3 B_3 \sin\alpha_3 \sin\beta_2 \quad (3.10)$$

$$T_i(5,4) = S_i(3,2) \quad \text{(nos. refer to subscripts)} \quad (3.11)$$

$$P'_i(C,B,A,\beta,\alpha) = P_i(C,A,B,\alpha,\beta) \quad (3.12)$$

$$Q'_i(C,B,A,\beta,\alpha) = Q_i(C,A,B,\alpha,\beta) \quad (3.13)$$

$$\epsilon_1 = P_2 T_1 - Q_2 S_1 \quad (3.14)$$

$$\epsilon_2 = P_2 T_2 + P_1 T_1 + P_4 T_1 - Q_2 S_2 - Q_1 S_1 - Q_4 S_1 \quad (3.15)$$

$$\epsilon_3 = P_1 T_2 + P_4 T_2 + P_3 T_1 - Q_1 S_2 - Q_4 S_2 - Q_3 S_1 \quad (3.16)$$

$$\epsilon_4 = P_3 T_2 - Q_3 S_2 \quad (3.17)$$

$$\eta_1 = P_2 T_3 - P_1 T_1 - Q_2 S_3 + Q_1 S_1 \quad (3.18)$$

$$\begin{aligned}\eta_2 = {} & P_2 T_4 + P_1 T_3 + P_4 T_3 + P_2 T_1 - P_3 T_1 - P_1 T_2 - Q_2 S_4 \\ & - Q_1 S_3 - Q_4 S_3 - Q_2 S_1 + Q_3 S_1 + Q_1 S_2 \end{aligned} \quad (3.19)$$

$$\begin{aligned}\eta_3 = {} & P_1 T_4 + P_4 T_4 + P_3 T_3 + P_4 T_1 + P_2 T_2 - P_3 T_2 - Q_1 S_4 \\ & - Q_4 S_4 - Q_3 S_3 - Q_4 S_1 - Q_2 S_2 + Q_3 S_2 \end{aligned} \quad (3.20)$$

$$\eta_4 = P_3 T_4 + P_4 T_2 - Q_3 S_4 - Q_4 S_2 \quad (3.21)$$

$$\omega_1 = -P_1 T_3 + Q_1 S_3 \quad (3.22)$$

$$\omega_2 = P_2 T_3 - P_3 T_3 - P_1 T_4 - Q_2 S_3 + Q_3 S_3 + Q_1 S_4 \quad (3.23)$$

$$\omega_3 = P_4 T_3 + P_2 T_4 - P_3 T_4 - Q_4 S_3 - Q_2 S_4 + Q_3 S_4 \quad (3.24)$$

$$\omega_4 = P_4 T_4 - Q_4 S_4 \quad (3.25)$$

$\epsilon'_i, \eta'_i, \omega'_i =$ same as $\epsilon_i, \eta_i, \omega_i$ except that T_2 and T_3, and S_2 and S_3 are interchanged; i.e.,

$$\epsilon'_i(S_3, T_3, S_2, T_2) = \epsilon_i(S_2, T_2, S_3, T_3) \quad (3.26)$$

$$F_1 = \epsilon_1 \tan^2\psi_s + \epsilon_2 \tan^2\psi_s + \epsilon_3 \tan\psi_s + \epsilon_4 \quad (3.27)$$

$$F_2 = \eta_1 \tan^3\psi_s + \eta_2 \tan^2\psi_s + \eta_3 \tan\psi_s + \eta_4 \quad (3.28)$$

$$F_3 = \omega_1 \tan^3\psi_s + \omega_2 \tan^2\psi_s + \omega_3 \tan\psi_s + \omega_4 \quad (3.29)$$

$$F'_i(\epsilon', \eta', \omega', \phi_s) = F_1(\epsilon, \eta, \omega, \psi_s) \quad (3.30)$$

Table 3. continued...

$$\tan\phi_s = \frac{1}{2F_1}\left[-F_2 \pm \sqrt{F_2^2 - 4F_1F_3}\right] \tag{3.31}$$

$$\tan\psi_s = \frac{1}{2F_1'}\left[-F_2' \pm \sqrt{F_2'^2 - 4F_1'F_3'}\right] \tag{3.32}$$

Solve Equations (3.31) and (3.22) simultaneously for ϕ_s, ψ_s. Avoid the following trivial (meaningless) solutions

$$(a) \quad \phi_s = -\frac{1}{2}(p_4 + p_5); \quad \psi_s = -\frac{1}{2}(q_4 + q_5) \tag{3.33}$$

$$(b) \quad \tan\phi_s = \left(\frac{T_3S_1 - T_1S_3}{T_1S_2 - T_2S_1}\right)\tan\psi_s + \left(\frac{T_4S_1 - T_1S_4}{T_1S_2 - T_2S_1}\right) \tag{3.34}$$

$$\omega_1 = (T_1S_4 + T_2S_3 - T_3S_2 - T_4S_1)/(T_1S_3 - T_3S_1) \tag{3.35}$$

$$\omega_2 = (T_2S_4 - T_4S_2)/(T_1S_3 - T_3S_1) \tag{3.36}$$

$$\tan\psi_s = -\frac{1}{2}\omega_1 \pm \frac{1}{2}\sqrt{\omega_1^2 - 4\omega_2} \tag{3.37}$$

Example. $y = \log_{10} x$, as before and precision points are to be at $x = 1$, 1.431, 2.307, 4.190, 8.577. Suppose $r_\phi = 6\frac{2}{3}^\circ/x$ and $r_\psi = 90^\circ/y$. Then $a = 1$, $b = -1.216$, $c = 2.800$, $d = -3.186$, $\psi_0 = -85°56'$, $\phi_0 = 192°41'$. $e = 0.37$ per cent at $x = 10$. The error probably could be reduced by an additional factor of 2 by optimum spacing of the precision points. The methods described by Svoboda [4] in the chapter on final adjustment of linkage constants, can be applied in further refinements of this linkage, which is shown in Fig. 4.

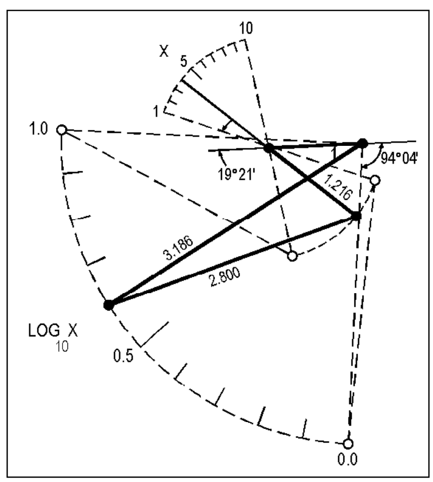

Fig. 4. Five-Point Approximation for Log Function.

Nth Order Approximations (Table 4)

In an nth order approximation, the ideal function is approximated by a Taylor power series up to and not including the nth derivative. The error is therefore roughly proportional to the nth derivative. The values of the derivatives are found upon differentiation of Equation (2), as per Table 4. Substitution of the ideal values of these derivatives into the basic Equations (4.1), Table 4, yields the conditions which determine the linkage.

Table 4. General Equations for an Nth-Order Approximation.

$$\left.\begin{matrix}\sin\phi_m\\ \cos\phi_m\end{matrix}\right\} R_1+(a_{i1}\sin\psi_m+a_{i2}\cos\psi_m)R_2=a_{i3}\sin(\phi_m-\psi_m)+a_{i4}\cos(\phi_m-\psi_m) \quad (4.1)$$

Use $\sin\phi_m$ for odd i; $\cos\phi_m$ for even i; $i=1,2,3,4,5,6$

$$R_1\cos\phi_m-R_2\cos\psi_m+R_3=\cos(\phi_m-\psi_m) \quad (4.2)$$

Coefficients a_{ij} in Equation (4.1)

a_{ij}	$i=1$	$i=2$	$i=3$	$i=4$
a_{i1}	$-m_1$	$-m_2$	$m_3-m_1^3$	$m_4-6m_1^2m_2$
a_{i2}	0	$-m_1^2$	$3m_1m_2$	$4m_1m_3-m_1^4+3m_2^2$
a_{i3}	$1-m_1$	$-m_2$	$m_3+(1-m_1)^2$	$m_4-6m_2(1-m_1)^2$
a_{i4}	0	$(1-m_1)^2$	$3m_2(1-m_1)$	$4m_3(1-m_1)+(1-m_1)^4-3m_2^2$

	$i=5$
a_{i1}	$-m_5+15m_1m_2^2+10m_1^2m_3-m_1^5$
a_{i2}	$-5m_1m_4+10m_1^3m_2-10m_2m_3$
a_{i3}	$-m_5-15(1-m_1)m_2^2+10(1-m_1)^2m_3+(1-m_1)^5$
a_{i4}	$-5m_4(1-m_1)+10m_2(1-m_1)^3+10m_2m_3$

	$i=6$
a_{i1}	$-m_6+15m_2^3+60m_1m_2m_3+15m_1^2m_4-15m_1^4m_2$
a_{i2}	$-6m_1m_5+45m_1^2m_2^2+20m_1^3m_3-15m_2m_4-10m_3^2-m_1^6$
a_{i3}	$-m_6+15m_2^3-60(1-m_1)m_2m_3+15(1-m_1)^2m_4-15(1-m_1)^4m_2$
a_{i4}	$-6m_5(1-m_1)-45(1-m_1)^2m_2^2+20(1-m_1)^3m_3+(1-m_1)^6+15m_2m_4+10m_3^2$

Error in ψ for a given value of ϕ:

$$\text{Error} \approx \frac{1}{n!}(\phi-\phi_m)^n m_n \quad (4.3)$$

Approximations obtained in this manner may not have the large range of the point approximations, but will have greater accuracy in the neighborhood of the precision point. The limiting order of approximation (like the limiting number of points) which can be handled without extremely lengthy computations, appears, in general, to be five.

5th Order Approximation (Table 5)

This approximation involves the solution of Equations (4.1) and (4.2) up to and including $i = 5$. r_ϕ and r_ψ are considered arbitrary. R_1 is eliminated using $i = 2, 4$ and also using $i = 3, 5$ in Equation (4.1); R_2 is eliminated from the resulting two expressions, yielding Equation (5.33). Solving for $R_1 \sin\phi_m$ from $i = 2, 4$, for $R_1 \cos\phi_m$ from $i = 3, 5$ in Equation (4.1), and dividing to obtain $\tan\phi_m$, an expression is obtained from which ϕ_m is eliminated by means of Equation (5.33), resulting in a 5th degree equation for $\tan\psi_m$, Equation (5.32). Knowing ϕ_m, ψ_m, Equations (4.1) and (4.2) can be solved as linear simultaneous equations in $R_{1,2,3}$.

Table 5. Fifth-order Approximation.

Linkage to generate $y = f(x)$ with precision point x_m, y_m and precision derivatives $m_i, i = 1, 2, 3, 4$. This involves solution of Equation (4.1), Table 4, up to $i = 4$, inclusive. See Table 4 for values of a_{ij}. Choose values for r_ϕ and r_ψ.

Let

$$f_1 = (a_{11} - a_{31})(a_{23} - a_{43}) - (a_{21} - a_{41})(a_{13} - a_{33}) \tag{5.1}$$

$$f_2 = (a_{11} - a_{31})(a_{24} - a_{44}) - (a_{21} - a_{41})(a_{14} - a_{34}) \tag{5.2}$$

$$f_3 = (a_{12} - a_{32})(a_{23} - a_{43}) - (a_{22} - a_{42})(a_{13} - a_{33}) \tag{5.3}$$

$$f_4 = (a_{12} - a_{32})(a_{24} - a_{44}) - (a_{22} - a_{42})(a_{14} - a_{34}) \tag{5.4}$$

$$g_1 = a_{31}a_{13} - a_{33}a_{11} \tag{5.5}$$

$$g_2 = a_{31}a_{14} - a_{34}a_{11} \tag{5.6}$$

$$g_3 = a_{32}a_{13} - a_{33}a_{12} \tag{5.7}$$

$$g_4 = a_{32}a_{14} - a_{34}a_{12} \tag{5.8}$$

$$h_i(a_{4i}, a_{2i}) = g_i(a_{3i}, a_{1i}), \quad i = 1, 2, 3, 4 \tag{5.9}$$

$$k_1 = a_{31} - a_{11} \tag{5.10}$$

Table 5 continued...

$$k_2 = a_{32} - a_{12} \tag{5.11}$$
$$l_1 = a_{41} - a_{21} \tag{5.12}$$
$$l_2 = a_{42} - a_{22} \tag{5.13}$$
$$\epsilon_1 = l_1 g_1 \tag{5.14}$$
$$\epsilon_2 = h_1 k_1 + l_1 g_3 + l_2 g_1 \tag{5.15}$$
$$\epsilon_3 = h_3 k_1 + h_1 k_2 + l_2 g_3 \tag{5.16}$$
$$\epsilon_4 = h_3 k_2 \tag{5.17}$$
$$\eta_1 = h_1 k_1 + l_1 g_2 \tag{5.18}$$
$$\eta_2 = k_1 h_2 + k_1 h_3 + k_2 h_1 - l_1 g_1 + l_1 g_4 + l_2 g_2 \tag{5.19}$$
$$\eta_3 = k_2 h_2 + k_1 h_4 + k_2 h_3 - l_2 g_1 - l_1 g_3 + l_2 g_4 \tag{5.20}$$
$$\eta_4 = k_2 h_4 - l_2 g_3 \tag{5.21}$$
$$\omega_1 = k_1 h_2 \tag{5.22}$$
$$\omega_2 = k_1 h_4 + k_2 h_2 - l_1 g_2 \tag{5.23}$$
$$\omega_3 = k_2 h_4 - l_1 g_4 - l_2 g_2 \tag{5.24}$$
$$\omega_4 = -l_2 g_4 \tag{5.25}$$
$$\lambda_1 = f_2^2 \epsilon_1 - f_1 f_2 \eta_1 + f_1^2 \omega_1 \tag{5.26}$$
$$\lambda_2 = 2 f_2 f_4 \epsilon_1 + f_2^2 \epsilon_2 - \eta_1 (f_2 f_3 + f_1 f_4) - f_1 f_2 \eta_2 + 2 f_1 f_3 \omega_1 + f_1^2 \omega_2 \tag{5.27}$$
$$\begin{aligned} \lambda_3 \;=\;& \epsilon_1 f_4^2 + 2 f_2 f_4 \epsilon_2 + f_2^2 \epsilon_3 - f_3 f_4 \eta_1 - \eta_2 (f_2 f_3 + f_1 f_4) \\ & - f_1 f_2 \eta_3 + \omega_1 f_3^2 + 2 f_1 f_3 \omega_2 + f_1^2 \omega_3 \end{aligned} \tag{5.28}$$
$$\begin{aligned} \lambda_4 \;=\;& f_4^2 \epsilon_2 + 2 f_2 f_4 \epsilon_3 + f_2^2 \epsilon_4 - f_3 f_4 \eta_2 - \eta_3 (f_3 f_2 + f_1 f_4) \\ & - f_1 f_2 \eta_4 + f_3^2 \omega_2 + 2 f_1 f_3 \omega_3 + f_1^2 \omega_4 \end{aligned} \tag{5.29}$$
$$\lambda_5 = f_4^2 \epsilon_3 + 2 f_2 f_4 \epsilon_4 - f_3 f_4 \eta_3 - \eta_4 (f_3 f_2 + f_1 f_4) + f_3^2 \omega_3 + 2 f_1 f_3 \omega_4 \tag{5.30}$$
$$\lambda_6 = f_4^2 \epsilon_4 - f_3 f_4 \eta_4 + f_3^2 \omega_4 \tag{5.31}$$
$$\begin{aligned} & \lambda_1 \tan^5 \psi_m + \lambda_2 \tan^4 \psi_m + \lambda_3 \tan^3 \psi_m + \lambda_4 \tan^2 \psi_m \\ & + \lambda_5 \tan \psi_m + \lambda_6 = 0 \end{aligned} \tag{5.32}$$
$$\tan(\phi_m - \psi_m) = - \left[\frac{f_2 \tan \psi_m + f_4}{f_1 \tan \psi_m + f_3} \right] \tag{5.33}$$
$$\text{Error in } \psi \text{ at given value of } \phi \approx \frac{1}{5!} (\phi - \phi_m)^5 m_5 \tag{5.34}$$

Avoid the following trivial and meaningless solutions

$$\tan \psi_m = -k_2 / k_1; \quad \tan \psi_m = -l_2 / l_1 \tag{5.35}$$

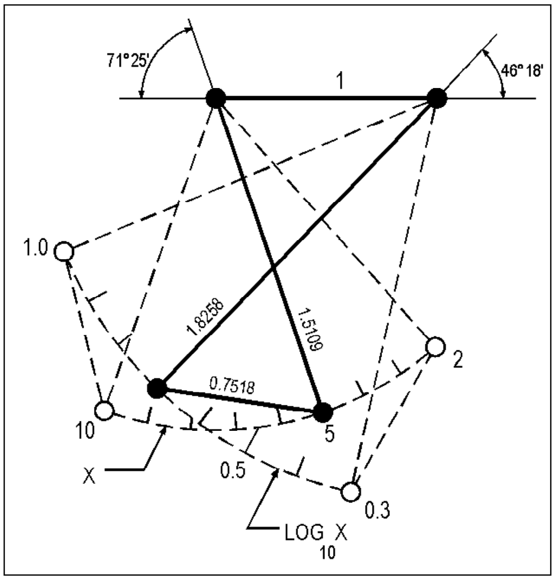

Fig. 5. Fifth-Order Approximation for Log Function.

Example. $y = \log_{10} x$, as before. Suppose $r_\psi \log_{10} e = 3/5$, $r_\phi x_m = 3/2$. Then $a = 1$, $b = 1.5109$, $c = 0.75182$, $d = 1.8258$, $\psi_0 = -101°38'$, $\phi_0 = 213°13'$. $e = \frac{1}{2}$ per cent at $x = 2$, error = 0.2 per cent at $x = 10$, the precision point is $x = 5$, and the range restricted to $2 \leq x \leq 10$. The linkage is shown in Fig. 5.

Higher-Order Approximations (Tables 6 and 7)

Such approximations can be determined if the function is simple enough.

Examples. To a 6th order approximation, the squaring linkage can be synthesized by letting $m_1 = \frac{1}{2}$, $R_2 = 1$, $\phi_m = 0$. The solution assumes a simple form, Table 6, in which m_2 is arbitrary. This gives rise to an infinite number of linkages. When $m_2 = \frac{1}{2}$, for instance, the linkage shown in Fig. 6 and Table 6 is obtained.

The tangent function can be represented to the 7th order at (0,0) by letting $\phi_m = \psi_m = 0$, at which point, however, the linkage assumes an indeterminate position. The solution obtained after considerable algebra, is shown in Table 7, in which λ is an arbitrary parameter. The particular case $\lambda = -0.6$ is shown in Table 7 and Fig. 7. The accuracies of this and the squaring linkages are noteworthy.

Table 6. Linkages Mechanizing the Squaring Function y = x^2 by Means of a 6th-Order Approximation.

$$m_1 = \frac{1}{2},\ m_2 \text{ arbitrary},\ x_m = 1/(2r_\phi m_2),\ \phi_m = 0. \tag{6.1}$$

$$\tan\psi_m = \frac{3+48m_2^2}{8m_2} \tag{6.2}$$

$$R_1 = 2m_2\sin\psi_m + \frac{1}{2}\cos\psi_m \tag{6.3}$$

$$R_2 = 1 \tag{6.4}$$

$$R_3 = 2\cos\psi_m - R_1 \tag{6.5}$$

In the particular case $m_2 = 1$

$$\psi^\circ = 73^\circ 55' 23'' + \frac{(\phi^\circ + 28.6479)^2}{114.592} \tag{6.6}$$

$$a = 1, b = 1, c = 1.9838, d = 0.48702 \tag{6.7}$$

e = error in ψ for given value of ϕ, expressed as % of max travel of ψ

$$e(\phi) = e(-\phi) \tag{6.8}$$

$$e(\phi = \pm 35^\circ) = 0.16\% \tag{6.9}$$

$$e(\phi = \pm 45^\circ) = 0.6\% \tag{6.10}$$

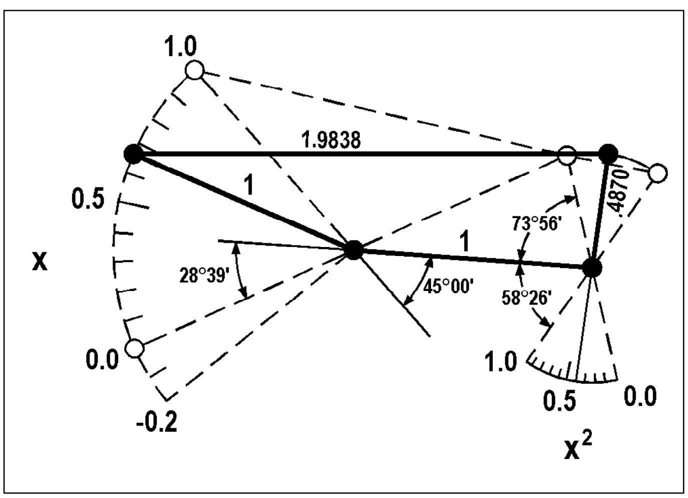

Fig. 6. Sixth-Order Approximation for Squaring Function.

Table 7. Linkages Mechanizing Tangent Function and Hyperbolic Tangent Function by Means of 7th Order Approximations.

The functions $y = \tan x$ and $y = \tan\mathrm{h}\, x$ are mechanized at $x = y = 0$ to the 7th order. The tangent function is obtained when α (Equation 7.4) is positive. The hyperbolic tangent function is obtained when α is negative.

$$\phi_m = \psi_m = 0; \quad \lambda \text{ is an arbitrary parameter.} \tag{7.1}$$

$$m_1 = \frac{5 - 60\lambda^2 - 80\lambda^3 \pm 3\sqrt{144\lambda^4 - 160\lambda^3 - 120\lambda^2 + 1}}{80\lambda^3 + 96\lambda^2 + 4} \tag{7.2}$$

$$\alpha = (m_1 - 1)(m_1 + 1)\lambda \tag{7.3}$$

$$m_3 = \alpha m_1 \tag{7.4}$$

$$m_5 = 4\alpha^2 m_1 \tag{7.5}$$

$$R_1 = (1 - m_1)^2 + m_1^2 R_2 \tag{7.6}$$

$$R_2 = \frac{4m_3(1 - m_1) + (1 - m_1)^4 - (1 - m_1)^2}{4m_3 m_1 - m_1^4 + m_1^2} \tag{7.7}$$

$$R_3 = 1 + R_2 - R_1 \tag{7.8}$$

In the particular case $\lambda = -0.6$

$$a = 1,\ b = 2.4286,\ c = 0.59524,\ d = 2.83333 \tag{7.9}$$

$$\psi^\circ = 60.395 \tan(\phi^\circ / 2.1082) \tag{7.10}$$

$$\text{Error, } e(x = 45^\circ) = \frac{1}{2}\%; \quad e(x = 42^\circ) = 0.1\% \tag{7.11}$$

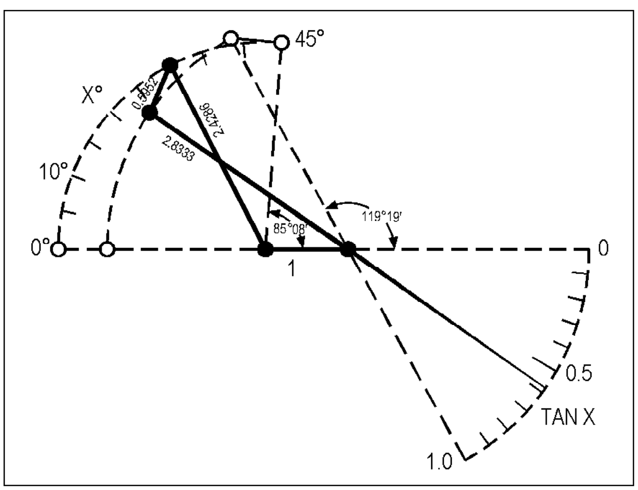

Fig. 7. Seventh-Order Approximation for Tangent Function.

Imaginary quantities often occur in the solutions of nth order approximations. In general, the real and imaginary parts then can be equated separately and physical meaning assigned to each of the resulting expressions. Thus, for example, in the case of the just considered tangent function, when α is negative, the scale factors become pure imaginaries. If $r_\phi = iR_\phi$, $r_\psi = iR_\psi$, the ideal relationship becomes

$$\frac{\psi - \psi_0}{iR_\psi} = \tanh\left(\frac{\phi - \phi_0}{iR_\phi}\right)$$

Equating real and imaginary parts, one obtains

$$\frac{\psi - \psi_0}{R_\psi} = \tanh\left(\frac{\phi - \phi_0}{R_\phi}\right)$$

In this case, the sign of one term in the power-series approximation for $\tan\phi$ has been reversed. For example, when λ approaches infinity, we have in the limiting case, $m_1 = -1$, $\alpha = -9/5$, $a = 1$, $b = -3/11$, $c = 25/11$, $d = 3$, and the ideal relationship becomes

$$\psi^\circ = -60.395 \tanh\left(\frac{\phi^\circ}{60.395}\right)$$

At $x = 1$, $e = 0.44$ per cent.

Conclusion

Analytical methods have been developed for the approximate generation of a function, using a 4-bar linkage having either several precision points or a single precision point with several precision derivatives. With careful choice of the arbitrary parameters, linkages can be synthesized in a systematic manner.

Acknowledgements

The author wishes to thank Prof. H. Dean Baker for his valuable advice and guidance, Prof. Francis J. Murray from whom the author has received much assistance, and the various members of the Mechanical Engineering Department of Columbia University, whose advice has been of considerable benefit. The work was undertaken in part with the aid of the Du Pont

fellowship in Mechanical Engineering for the years 1951–1952, and 1952–1953.

Discussion

A.E. Richard De Jonge [7]. The paper is the companion paper to the first paper by the author [1], which was discussed comprehensively by the writer. It is unfortunate that both the presentation and the printing of the present paper have been so long delayed by the Society.

As to terminology, similar remarks to those made by the writer in the discussion of the author's first paper apply.

The paper is intended to make the construction of 4-link mechanisms for the functional relationships for computing purposes relatively easy and certain. This holds for matching shaft rotations of links in a 4-link mechanism. Attempts to obtain a similar result have been made in this country by Svoboda and others, which have been much publicized. However, the fact should not be lost sight of that other attempts also have been made abroad. In Germany, K.–H. Sieker [8] has used the Euler–Savary equation and a modification of it by him to obtain "order" type approximations. His method, however, is not as extensive or as powerful as that given in the author's paper. Kurt Hain [9], in Germany, has used his method of reducing point poisitions (Punktlagenreduktion) for obtaining approximations of given functions by 4-link mechanisms. His method, too, is not as powerful as that of the author. The thought occurs to the writer that, by combining Sieker's and Hain's methods, a simple and useful method may be evolved. In Russia, attempts to represent functions by mechanisms were made almost a century ago by P. L. Chebychev [10] who used for this purpose his method of, what is now called, Chebychev polynomials. He and his present-day followers have progressed further along this path, but their methods are not as simple to use or as powerful as those presented in the author's paper, which are more systematic.

In the present paper, extensive use has been made of the developments given in the author's first paper, although this may not appear on the surface, because it is hidden in Equation (2) on which the formulas presented are based. The latter are explained only in a general way, but lack concise derivation. This is due to the great amount of space that would be required to present the full derivations, a work that has taken the author several years to complete. It is a great pity that, on account of lack of available space, this important work could not be made available to the profession.

The principle of the paper's methods is simple. It is based on 4-link mechanisms, the rotations of two links of which are to be matched so as to conform to a given function. The fundamental relation is given by Equation (2) which is the simplest form obtainable between the angles. Thus the method is easy to understand in principle and, as far as getting acquainted with its use is concerned, it is for the average engineer, who is not a kinematician, obviously simpler to acquire than the European geometrical methods which necessitate intensive study, very accurate draftsmanship, and a good deal of experience before they can be applied with satisfactory results.

Naturally, there are advantages and disadvantages in every method, and the author's method is no exception to this rule. The principal advantage to the American trained engineer, who has a predominantly analytical training, is that it is an analytical method. Furthermore, it is exact; that is, does not require trial-and-error attempts. On the other hand, a great disadvantage is that no geometrical picture is given by the method, so that ranges and proportions are not constantly in view. Thus no immediate check is possible of whether the results may get out of hand, which is the advantage of the geometrical methods and cannot be achieved by an analytical method. This raises the question, alluded to before, whether it might not be possible to combine both analytical and geometrical methods to mutual advantage.

Equation (2), which is fundamental for the author's method, was derived by him in the discussion of a previously published paper [11]. The author's method operates with the so-called precision points which are those points in which the function to be generated (ideal function) and that produced by the generating mechanism (actual function) coincide. Two types of approximation are dealt with. In the first, a number of precision points (3,4, $\ldots$ n) are chosen. These approximations are called third, fourth, $\ldots$ nth order approximations. To determine them, the original Equation (2) is used as well as its consecutive derivatives. Thus sufficient equations are available to make the two functions coincide at the precision points. Between these, greater or less deviations occur. The second type is that in which only one precision point is chosen, but at that point the derivatives also are made to coincide up to the order desired.

The paper, then, describes briefly the various orders of approximation up to the nth order and gives tables of the equations required up to the seventh order, at least for particular functions.

The greatest benefit of the use of the author's method, naturally, is obtained when computing machines can be utilized. Especially the punched-card digital computers lend themselves well to the computations that have to be effected. Thus a great deal of labor will be saved.

The methods described are powerful and systematic, and are based on a simple idea originally derived as a result of the symmetrical parameter P, as derived in the author's first paper. By means of this parameter, Equation (2) is very simply derived although in a different manner than in the present paper.

As far as is known, the author's method accomplishes more than either the German or the Russian methods have achieved hitherto, although both these latter are more or less equivalent in the cases of the lower order approximations, say, up to five precision points.

The choice of the precision points in the author's and the German methods is arbitrary, and the best choice is a matter of experience and intelligent estimation. The Russian Chebychev method is difficult to apply, but when it can be applied, the best positions of the precision points are automatically selected; that is, the precision points are so chosen that the maximum error is minimized.

Since the general field of linkage synthesis is receiving attention both here and abroad, this paper should be of interest not only to engineers in the Uniter States, but also to those in other countries, and it should make them realize that they are not the only ones who can boast progress in this field. In fact, this present paper by the author goes further than either the Germans or the Russians have gone so far. The paper should, therefore, arouse great interest both here and abroad.

The paper has attempted to utilize the rotations of two links of a 4-link mechanism. It seems to the writer that this is only a first step in the desired direction. He feels that coupler curves may be made use of to advantage. Inasmuch as there are among these many whose arc approximates a circle over a rather wide range, and inasmuch as it is rather easy to determine their deviations from a circle by means of their evolutes, they may be used to obtain rotations of an additional crank linking the describing point to the frame. This would then be no longer a 4-link mechanism, but rather a redundantly closed 5-link mechanism. However, it seems that better approximations may be achieved thereby. No attempt as yet has been made to exploit this idea which the writer offers here as a further avenue of approach. In addition, this method should have the desired advantage of linking together geometrical and analytical methods which, in itself, would be a noteworthy achievement.

To sum up, the author's paper is not only a very interesting, but also a very timely one of great practical consequence. For this important contribution to the synthesis of mechanisms to generate desired functions with great approximation, the author deserves high praise.

A.S. Hall Jr [12]. It seems to the writer that the author has succeeded in reducing the analytical approach to the 4-bar problem to as uncomplicated a form as is possible. In so doing he has shown that the analytical treatment may not be any more difficult or time-consuming than the older graphical-geometric methods. The analytical treatment will appeal to many because it fits in better with their mathematical backgrounds. Upon encountering for the first time a problem of the type discussed in this paper most graduates of American engineering schools could probably proceed more quickly by following the author's lead than by familiarizing themselves with the work of Burmester and his successors.

The author has brought together in a unified treatment the problem of approximating a function at a finite number of points and the problem of approximating a function plus a finite number of its derivatives at a single point. Speaking as a teacher, the writer feels that this is an important feature of the paper. The intimate relation between the two problems might be emphasized further by noting that an nth order approximation is an n-

point approximation in which the n-points are infinitesimally near.

The author is to be congratulated on an important contribution to kinematic synthesis, not only for the problems he has solved, but also for the encouragement his work gives to attack other problems by similar techniques.

R.L. Kenngott [13]. To mechanize an arbitrary function by means of a 4-bar linkage, the designer would like to employ a simple, straightforward process whereby he might insert the characteristics of the given function, "turn the crank", and have the optimum linkage parameters fall out automatically and in due conformity with all practical considerations such as favorable mechanical advantage, manufacturing tolerances, and so on. Lacking this, he has been wont to exercise a more or less intelligent guess as to a probably suitable linkage, thereupon to refine the function-fit by one or another species of "cut-and-try" with varying degrees of power and precision in converging to an optimum or, at least, a tolerable solution. However, there is always a suspicion that his initial guess has unnecessarily restricted his solution to one and, perhaps, not the best of several possible solutions; for there exist eight distinct and discontinuously related classes of functions generated by the 4-bar linkage (two of these being generated by the familiar drag-link and the crank-and-rocker, for example). That is to say, convergence to a solution by variation of the parameters is apt to confine itself to that single class of 4-bar linkage function determined by the initial guess. Consequently, the designer has need of two kinds of tools, a ready means for exploring the potential field of 4-bar linkage functions in order to get off to a good start and a precise means for refining the function-fit to within a prescribed tolerance.

Clearly, this paper represents a long stride in the direction of taking the guesswork out of linkage synthesis. Unfortunately, it seems inherent in linkage systems that the complexity of design computation must increase as the guesswork is reduced; so that his most powerful procedures, for 5-point and for n-th order approximations, are likewise the most abstruse. Moreover, it might be worth noting that nth order approximation is useful in mechanizing only fairly well-known analytic functions in that it requires evaluation of the first $n-1$ derivatives of the ideal function. Calculation of derivatives higher than the second is seldom meaningful with empirical functions unless resort is first had to their approximation by a polynomial, an harmonic expansion, or other familiar analytic expression. Now, in this writer's experience, those designers who are not literally steeped in the linkage art, which is to say most designers of computing linkwork, will prefer and use universally any one simple method of synthesizing a linkage to the exclusion of various complex methods however powerful – even if the simple method be limited in scope and involve considerable drudgery.

It is in just this respect that the writer believes the author has made a most valuable contribution. His 3-point method of approximate synthesis is simple, readily understood, and, above all, quickly and easily manipulated. It should provide good first approximation and even a tolerable final approximation to a wealth of arbitrary functions, particularly those whose graphs display curvature of constant sign. For, if the range of both input and output motions be limited to about 120 deg, which is generally most practical, no 4-bar linkage will generate a function with appreciable inflection in curvature, and hence no extreme errors can be encountered. On the other hand, where the ideal function itself possesses an inflection, a 4 or 5-point approximation may be required. However, this is likely to be seldom; and the 4 and 5-point methods are not so distantly related to the more generally useful 3-point method as to discourage the designer altogether once he has become familiar with the latter.

In any event, the author's 3-point approximation should serve well to explore rapidly the field of potentially suitable 4-bar linkage functions, a valuable first step before refinement of the function-fit becomes worth while. For the rest, he has probed deeply into the analytical approach to design of 4-bar linkages and may well lead to the ultimate goal of a simple, universal method of synthesis with guesswork entirely eliminated.

P.T. Nickson [14]. The design or synthesis of linkages is generally a problem of choosing a set of arbitrary parameters, so that the resulting linkage configuration produces a desired motion. Several methods are available to the designer. These methods may involve graphical solutions, trial-and-error approaches, selection of variables from systematically arranged tabular data or mathematical solutions. None of these methods produce exact solutions, but only give approximations of the given conditions. It may be required that the approximations be nearly exact at only a few points in the motion, or at several points in the motion. The complexity of the problem then, is dependent upon to the number of points that must be satisfied and the accuracy of the approximation. For more accurate solutions, it often becomes necessary to add refinements to these methods.

The author presents a very interesting refinement in linage synthesis. His treatment is a mathematical one and involves the solution of several simultaneous equations. These equations are derived from expressions defining the function to be generated and expressions relating the various linkage parameters. Methods are shown for 3-point, 4-point, 5-point, and higher-order approximations. Each method is illustrated with numerical examples, including sketches of the resulting linkages. His method is applied to the synthesis of 4-bar linkages; however, it is equally applicable to other types of linkage mechanisms.

In order to use this method, the designer must have a sufficient mathematical background. If this knowledge is limited, as is generally the case, he must turn to other less refined methods. Then accuracy or exactness of the approximation is achieved only by repetitive solutions, with the point of diminishing returns very quickly reached. The author's method is a fine tool for the mathematician but it should be avoided by those not so gifted.

We have had excellent results with the so-called "overlay method". This method is a graphical one and, for more cases, results in a quick and easy solution. If extreme care and accurate drawing techniques are practised, often very exact approximations are obtained. This method has the advantage, in that the geometrical relationships of the linkage members are easily seen. Thus dead-center positions and large angular relationships between links and slides may be avoided quite readily. Mathematical methods suffer in this respect, in that it is almost impossible to "picture" these relationships.

The designer, through practice and experience, will usually apply some one favorite method. He finds that this suffices his ordinary problems. Through diligence and good fortune, it may solve the occasional difficult synthesis. However, there always comes the problem wherein the usual practices are not adequate. It is here that help is required. What is needed are methods that improve the accuracy of the approximate solutions, obtainable by the simpler methods. In other words, after a set of parameters is established, higher accuracy should be obtainable by a slight modification of these values. Perhaps in future papers the author could use this approach to linkage synthesis. This is not meant in any way to detract from his fine paper, but these modification methods are invaluable to the linkage designer. The approximate synthesis is relatively easy; it is the narrowing down of the accuracy that really presents the problem.

Author's Closure

The light shed on the subject of linkage synthesis by the comments of the discussers adds appreciably to this investigation.

The point raised by Mr. de Jonge concerning the combination of a geometrical and an analytical technique (e.g., Hain-Sieker) is well taken. Some of the results of the present investigation can be used in this manner, as pointed out in the discussion of Mr. Kenngott, for instance. The suggestion to utilize one or more links attached to a suitable point on the coupler (connecting link) for the synthesis of functions difficult to mechanize by more elementary methods is a good one, although it is believed that present techniques are insufficient for such a refinement. Lack of space prevented the author from mentioning the noteworthy contributions of Alt, Artobolevskii, Beyer, Blokh, Chebichev, Hain, Sieker, and others with which the author is familiar. These investigators have tackled similar problems and in some cases have produced equivalent results. It is appreciated that Mr. de Jonge, whose writings have been instrumental in creating a revival of interest in the field of kinematics in this country, has called attention to the most significant of these contributions.

Professor Hall's comments are well taken and his comments relative to the essential equivalence of point and order approximations are borne out by a recent treatment of the subject of kinematic synthesis by R. Beyer [15].

The comments of Mr. Kenngott derive added significance in view of the discusser's association with A. Svoboda during the war days. The author agrees with Mr. Kenngott and the other discussers that the methods evolved represent only a beginning toward the ultimate goal of a simple, certain, and universally applicable method.

In connection with the valuable comments of Mr. Nickson, the author feels that difficulty is experienced not only in the refinement of the function, but also in the approximate synthesis (the initial fit). The refinement has, for instance, been treated in detail in Chapter 7 of Svoboda [4], further discussed in a paper by Hall and Tao [16], and can be analyzed by the theory of Chebichev, which states quite definitely what type of approximation can be expected to attain the least maximum error. It is the author's experience that the 3-point and the 3rd-order approximations are easier and faster to apply than the overlay method and other graphical techniques, especially by those not so gifted geometrically or analytically. The contrast between the methods increases when consideration is given to the use of automatic methods of computation, as suggested in the discussion by Mr. de Jonge. The author agrees with Mr. Nickson that all analytical methods suffer from the inability to provide a physical picture of the mechanism until after the completion of computations and that in this connection it would undoubtedly be desirable to strive for a suitable combination of geometrical and analytical techniques.

In conclusion, the author would like to express his appreciation of the interest shown by all the discussers. The beauty and complexity of linked mechanisms probably will continue to puzzle and fascinate us for generations to come.

Notes

[1] Analysis of four-bar linkages has been considered in a previous paper by the author, entitled "An Analytical Approach to the Design of Four-Link Mechanisms'', *Trans. ASME*, Vol. 76, 1954, pp. 483–492.

[2] Based on a thesis undertaken in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Mechanical Engineering at Columbia University, New York, N.Y.

[3] Assistant Professor, Department of Mechanical Engineering, Columbia University. Assoc. Mem. ASME.

Contributed by the Machine Design Division and presented at a joint session of the Machine Design Division and Lubrication Activity at the Fall Meeting, Milwaukee, Wis., September 8–10, 1954, of The American Society of Mechanical Engineers.

Note. Statements and opinions advanced in papers are to be understood as individual expressions of their authors and not those of the Society. Manuscript recived at ASME Headquarters, March 30, 1953. Paper No. 54–F-14.

[4] A Svoboda, *Computing Mechanisms and Linkages*, M. I. T. Radiation Series, McGraw-Hill Book Company, Inc., New York, NY, first edition, Vol. 27, 1948.

[5] E W Pike, T R Silverberg and P T Nickson, Linkage Layout, *Machine Design*, Vol. 23, pp.105–110, 194, November, 1951.

[6] J A Hrones and G L Nelson, *Analysis of the Four Bar Linkage*, The Technology Press of M.I.T. and John Wiley & Sons, Inc., New York, NY, 1951.

[7] Mechanical Engineer and Consultant, Reeves Instrument Corp., New York, N.Y. Mem. ASME.

[8] K.-H. Sieker, Ermittlung von Gelenkvierecken aus den Krümungshalbmessern der Polbahnen und deren Anderungen, *Die Technik*, Vol. 3, pp. 170–174, 1948.

[9] Kurt Hain, *Angewandte Getriebelehre*, Hermannn Schroedel Verlag, 1952.

[10] P L Chebychev, *Oevres* (Works, in French), Imperial Academy of Sciences, St. Petersburg, Russia, Vol.1, 1899; Vol.2, 1907.

[11] B W Schaffer and I Cochin, Synthesis of Four-Bar Mechanisms When the Position of Two Members Is Prescribed, *Trans. ASME*, Vol.76, p.141, 1954.

[12] Professor of Mechanical Engineering, Purdue University, Lafayette, Ind. Mem. ASME.

[13] Consultant in Design of Automatic Computers, Pleasantville, N.Y.

[14] Equipment Engineering Division, Raytheon Manufacturing Company, Newton, Mass.

[15] R Beyer, *Kinematische Getriebsynthese*, Verlag-Springer, Berlin, Germany, 1953.

[16] A S Hall and D C Tao, Linkage Design – A Note on One Method, *Trans. ASME*, Vol. 76, pp. 633–637, 1954.