In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

Counting Forumula For 3 × 3 Generalized Magic Squares

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The aim of this article is to explicitly compute the number of generalized magic squares of order 3. Counting functions for (generalized) magic squares of order n with fixed line sum r is known to be a polynomial function in r over $\mathbb{Q}$ of degree $(n-1)^2$. For higher values of n, it is quite difficult to explicitly write down such counting functions.

Introduction

The problem of finding the solutions of a system of linear equations over non-negative integers is of great importance. An $n \times n$ matrix over $\mathbb{N}$ whose line sums are constant is known as an *integer stochastic matrix* or *a magic square*. Here $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of non-negative integers and a line stands for either a row or a column. Let $H_n(r)$ be the number of $n \times n$ matrices over $\mathbb{N}$ having line sum r. For $n = 1$, we have $H_1(r) = 1$. For $n = 2$, one can see that every magic square with line sum r is of the form

$$\begin{bmatrix} r-s & s \\ s & r-s \end{bmatrix},$$

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It was shown by Ehrhart and Stanley that $H_n(r)$ is a polynomial function in r of degree $(n-1)^2$.

where $0 \leq \ s \leq \ r$ and we have $H_2(r) = r + 1$. It was shown by MacMohan [1] that

$$H_3(r) = \left(\begin{array}{c} r+4 \\ 4 \end{array} \right) + \left(\begin{array}{c} r+3 \\ 4 \end{array} \right) + \left(\begin{array}{c} r+2 \\ 4 \end{array} \right).$$

The first structural result for general n was proved by Ehrhart [2] and Stanley [3] in 1973. They showed that $H_n(r)$ is a polynomial function in r of degree $(n-1)^2$ that satisfies the identities

$$\begin{aligned} H_n(-1) = H_n(-2) = \ldots = H_n(-n+1) &= 0, \\ H_n(-n-r) = (-1)^{n-1}H_n(r). \end{aligned}$$

That the counting function $H_n(r)$ is a polynomial in r of degree $(n-1)^2$ over $\mathbb{Q}$ was conjectured earlier by Anand, Dumir and Gupta [4].

Generalized Magic Squares

For a non-negative integer m, let A be an $n \times n$ matrix over $\mathbb{N}$ having ith line sum $i^m r$, where by ith line we mean ith row or column. For $m = 0$, we get a magic square. We shall call such matrices *generalized magic squares*. For a fixed $m > 0$, let

$$H_n^m(r) := \text{number of } n \times n \text{ matrices over } \mathbb{N} \text{ with } i\text{th line sum } i^m r.$$

It was proved by Stanley [5] that $H_n^m(r)$ is a polynomial function in r over $\mathbb{Q}$. Keeping n fixed, we see that for $n = 1$, $H_1^m(r) = 1$ and $H_2^m(r) = r + 1$. Our aim is to find a formula for the counting function $H_3^m(r)$. In order to count all 3×3 generalized magic squares, we shall actually write all of them. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

be a generalized magic square, i.e., $a_{ij} \in \mathbb{N}$ and $\sum_{s=1}^{3} a_{is} = i^m r = \sum_{t=1}^{3} a_{ti}$. Since the sum of the first row or first column is r, we have $a_{11} = r - s$, where $0 \leq s \leq r$ and for each value of $a_{11} = r - s$ other entries of first row and first column are $(a_{12}, a_{13}) = (j, s-j)$ and $(a_{21}, a_{31}) = (k, s-k)$, where $0 \leq j, k \leq s$. Since the sum of second row or second column is $2^m r$, putting $a_{22} = t$, we have $a_{23} = 2^m r - k - t$, $a_{32} = 2^m r - j - t$, where $0 \leq t \leq \min\{2^m r - j, 2^m r - k\}$. Also the sum of third row or third column is $3^m r$, $a_{33} = (3^m - 2^m)r + j + k + t - s$ and as $m \geq 1$, we have $(3^m - 2^m)r + j + k + t - s \geq (r - s) + j + k + t \geq 0$. Thus for $a_{11} = r - s$, every 3×3 matrix over $\mathbb{N}$ with ith line sum $i^m r$ is of the form

$$\begin{bmatrix} r-s & j & s-j \\ k & t & 2^m r - t - k \\ s-k & 2^m r - t - j & (3^m - 2^m)r + j + k + t - s \end{bmatrix},$$

where $0 \leq j, k \leq s$ and $0 \leq t \leq \min\{2^m r - j, 2^m r - k\}$. Note that $\min\{2^m r - j, 2^m r - k\} = 2^m r - l$, where $0 \leq l \leq s$. Now for a fixed value of l, we need to find the values of j and k for which $\min\{2^m r - j, 2^m r - k\} = 2^m r - l$. There are exactly two cases, either $j = l$ and $0 \leq k \leq j = l$, or $k = l$ and $0 \leq j \leq k = l$. Thus there are $2l + 1$ pairs of j and k for which $\min\{2^m r - j, 2^m r - k\} = 2^m r - l$. Since $0 \leq t \leq \min\{2^m r - j, 2^m r - k\} = 2^m r - l$ and for each such t there are exactly $2l + 1$ pairs of j and k, the number of such matrices is $(2l + 1)(2^m r - l + 1)$. Thus the total number of 3×3 generalized magic squares with $a_{11} = r - s$ is

$$\sum_{l=0}^{s} (2l + 1)(2^m r - l + 1).$$

Since $a_{11} = r - s$, where $0 \leq s \leq r$, the number of 3×3

generalized magic squares is given by

$$\begin{aligned} H_3^m(r) &= \sum_{s=0}^{r}\sum_{l=0}^{s}(2l+1)(2^m r - l + 1) \\ &= \frac{(r+1)(r+2)}{6}\left[(2^{m+1}-1)r^2 + 3(2^m)r + 3\right]. \end{aligned}$$

Thus we have a counting formula for 3×3 generalized magic squares.

Now it would be interesting to count the number of 3×3 *symmetric* generalized magic squares. As in the generalized magic square case, one can see that a $3\times$ symmetric generalized magic square with ith line sum $i^m r$ is given by

$$\begin{bmatrix} r-s & j & s-j \\ j & t & 2^m r - t - j \\ s-j & 2^m r - t - j & (3^m - 2^m)r + 2j + t - s \end{bmatrix},$$

where $0 \le s \le r$, $0 \le j \le s$ and $0 \le t \le 2^m r - j$. Let $S_3^m(r)$ denote the number of 3×3 symmetric generalized magic squares with ith line sum $i^m r$. Then

$$S_3^m(r) = \sum_{s=0}^{r}\sum_{j=0}^{s}\sum_{t=0}^{(2^m r - j)} 1 = \sum_{s=0}^{r}\sum_{j=0}^{s}(2^m r + 1 - j),$$

which on simplification gives

$$S_3^m(r) = \frac{(2^m r+1)(r+1)(r+2)}{2} - \frac{r(r+1)(r+2)}{6}.$$

This is a counting formula for 3×3 symmetric generalized magic squares.

Suggested Reading

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