Mirrors and Merry-Go-Rounds

Joseph Samuel

This is an elementary introduction to rotations in three dimensions, using reflections to naturally introduce spinors. It provides a stepping stone to higher mathematics and some new perspectives.

1. Introduction

If you look around you, you will see rotations everywhere. Wheels on cars, fans, spin on cricket balls, giant wheels, merry-go-rounds.... Your head rotates when you look around. The Earth rotates and it's a very good thing that it does! But for the rotation of the Earth, our days would be long indeed and our lives correspondingly short! Some of us would get roasted and the others frozen. The Sun rotates, as do black holes, the solar system, the galaxy and our local cluster of galaxies. Looking down in scale instead of up at the sky, some bacteria have rotary engines to propel themselves. Molecules rotate. Electrons and many other elementary particles have spin. Nuclei have spin and this leads to a life-saving medical probe: Magnetic Resonance Imaging (MRI).

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Scientists encounter rotations in many areas of research. In physics we deal with the mechanics of rotations (of galaxies, stars, black holes, nucleii or elementary particles) and with the mathematics of rotations. Mechanics is a wide term including classical mechanics, quantum mechanics and statistical mechanics and rotation is studied in all three branches of mechanics. This article will focus on some mathematical aspects of rotations, trying to understand the 'space of rotations'. As you will see, reflecting on rotations can make your head spin! Prerequisites for following this article are a familiarity with complex analysis, trigonometry, vector

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calculus, google to look up unfamiliar terms and most importantly some paper and pencils to supply missing steps.

2. Topology

Here is a simple experiment which you can do: take a glass of water in the palm of your hand and, keeping it upright (so the water doesn't spill) rotate it by 360° anticlockwise (2π in radians). You will find that your arm is twisted uncomfortably. Continue the anticlockwise rotation and you find that your arm untwists after 720° (4π radians)![1] There is a sense in which 4π rotation is trivial but 2π is not. Understanding this simple experiment leads us naturally to topology and higher mathematics.

[1] Another example is the Dirac Belt trick, which has the same intellectual content. See http://www.gregegan.net/APPLETS/21/21.html

Let's now set ourselves to understand rotations in three dimensions. Consider ordinary three-dimensional space and fix an origin so that the coordinates of points in space are described by three real numbers (x, y, z). These form a vector and we will sometimes denote it by $\vec{r}$. Rotations are linear transformations of space that preserve the lengths of vectors: $\vec{r}.\vec{r} = x^2 + y^2 + z^2$ is invariant under the transformations. But not all linear transformations that preserve length are rotations. Reflections also preserve length, as you can see in a plane mirror. Reflections are relatively clean operations – they reverse some components of a vector and preserve others. The main point of this article is that two reflections lead to a rotation and so rotations can be broken up into reflections. Reflections are in some ways simpler than rotations. (Mirrors are "silver and exact" to borrow words from the poet Sylvia Plath!) We will use reflections to understand rotations. The turning of a giant wheel or a merry-go-round at a fair can be understood from quite abstract points of view. One would not have expected higher mathematics to lurk in such a commonplace phenomenon as rotation.

Rotations can be broken up into reflections.

Consider linear transformations of $\mathbb{R}^3 (\vec{r} \in \mathbb{R}^3)$

$$\vec{r}\,' = R\vec{r}, \tag{1}$$

which preserve length:

$$\vec{r}\,' \cdot \vec{r}\,' = \vec{r} \cdot \vec{r}. \tag{2}$$

From (2) it follows that R satisfies

$$R^{\mathrm{T}} R = 1, \tag{3}$$

where R^{T} is the matrix transpose of R. We say that R is an orthogonal matrix. It also follows by considering $\vec{r}(\lambda) = \vec{r}_1 + \lambda \vec{r}_2$, (where λ is an arbitrary real number), and its transformation, that R preserves inner products (and therefore angles) between vectors

$$\vec{r}_1\,' \cdot \vec{r}_2\,' = \vec{r}_1 \cdot \vec{r}_2. \tag{4}$$

From (3) it follows that $(\det R)^2 = 1$ or $\det R = \pm 1$. Transformations with $\det R = +1$ are called rotations. These preserve the handedness of frames. Those transformations with $\det R = -1$ are called reflections. These *reverse* the handedness of frames and are called *improper*. The transformations which preserve length form a group called $O(3)$, which includes rotations as well as reflections. (O means orthogonal (see equation (3)). The 3 in $O(3)$ tells us the dimension we are in, which is three dimensions. Rotations form a subgroup called $SO(3)$. (S stands for special, which means $\det R = 1$).

Euler studied rigid body motion and in the process anticipated spinors.

Box 1. Leonhard Euler (1707–1783)

Euler had many theorems, one has to be specific about *which* theorem of Euler one is referring to. Apart from the theorem on rotations used in the text, Euler has theorems in several branches of mathematics and physics, from number theory to fluid mechanics. He had phenomenal powers of concentration and could work while dandling a child on his knee! He left behind a stack of manuscripts which kept mathematicians busy till recently – sorting, annotating and publishing them. They have now given up on this task and are just scanning the rest of Euler's manuscripts and uploading them on the web.

Euler's theorem on rotations (see *Box* 1) states that every rotation leaves some direction invariant. Euler's theorem can be proved by noting that over the complex numbers the characteristic equation

$$\det(R - \lambda I) = 0$$

of R is a cubic polynomial with real coefficients. From (3), (re-written as $R^\dagger R = 1$, since R is real), we see that

We want a 'map' of the space of rotations. Such a map has to be three dimensional.

R is unitary, which implies $|\lambda| = 1$ and the eigenvalues lie on the unit circle. Since the coefficients of the characteristic equation are real, if λ is an eigenvalue, so is its complex conjugate $\bar{\lambda}$. The three roots must be of the form $(e^{i\theta}, e^{-i\theta}, 1)$ since their product has to be unity from $\det R = 1$. The eigenvector corresponding to $\lambda = 1$ is the direction that is left invariant by the rotation. This proves Euler's theorem: Every rotation in three dimensions has an axis $\hat{n}$ which is unchanged by the rotation. (Note that Euler's theorem is true in *all* odd dimensions and *not true* in *all* even dimensions).

What are all the rotations? In other words what is the space of rotations? We want to find a space so that each point of the space corresponds to a rotation and each rotation is represented by one point. We know that rotations are characterised by an axis $\hat{n}$ (which needs two numbers to specify it) and an angle θ which needs one number. The space of rotations is three-dimensional. So, we will not be able to represent the space of rotations on a sheet of paper, but we can use points of three-dimensional space. We use the tip of the vector

$$\vec{V}_R = \theta\hat{n}$$

to represent a rotation. If $\theta = 0$, $\vec{V}_R = 0$, and the rotation by zero about any axis is represented at the origin. This is the identity of $SO(3)$. Rotations by non-zero angles are represented by non-zero vectors in the direction of the axis. What is the range of θ? We remember that rotation by π about $\hat{n}$ is the same as rotation by π about $-\hat{n}$. Thus, θ goes from 0 to π. The picture of the rotation group $SO(3)$ looks like a solid ball. The centre of the ball is the identity. (We must remember though that adding two $\vec{V}$ vectors doesn't make sense. The solid ball is just a representation of the rotations. To compose rotations we have to do more work. Simply adding the corresponding vectors is definitely wrong, because rotations don't commute!) Although the solid

The space of rotations is a solid ball with antipodal points identified.

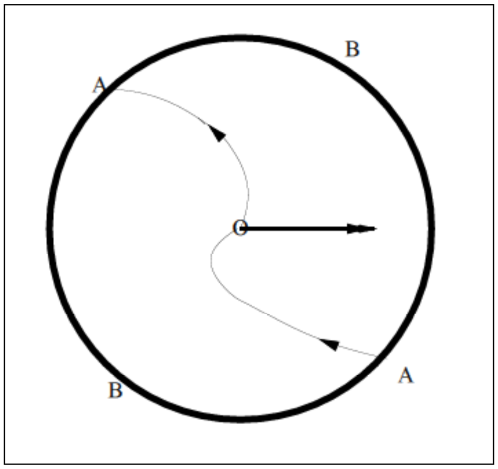

Figure 1. The space of rotations is a sphere with opposite points identified (declared to be the same or glued together). The center of the sphere is marked O and represents the zero rotation. The vector shows a particular rotation whose direction is the axis of rotation and whose length represents the angle. The radius of the sphere is π. Also shown is a curve that starts from O, goes to A at the edge of the sphere and returns to O from the antipodal point, which is also labelled A. This is a closed curve in the space of rotations which cannot be continuously shrunk to a point.

ball has a boundary, we must remember that opposite points on its surface represent the same rotation.

This last remark leads to an interesting observation. There are closed curves in $SO(3)$ that start from the origin, go to a point $V_R = \pi\hat{n}$ at $\theta = \pi$ and return from its antipode $-\pi\hat{n}$ to the origin (*Figure* 1). The total angle of rotation as one traverses this curve is 2π. Such curves cannot be shrunk to a point. We say that $SO(3)$ is 'multiply connected'.

However, transversing this loop twice (rotating by 4π) leads to a curve which can be shrunk to a point! 2π rotations are topologically non-trivial. 4π rotations are topologically trivial. This is the mathematics behind the simple experiment with a glass of water that we started with.

$SO(3)$ can be represented as a ball with opposite points of the boundary identified, i.e., to be regarded as the same point. Equivalently, consider a sphere in four dimensions (x_1, x_2, x_3, x_4) all real with $x_1^2+x_2^2+x_3^2+x_4^2 = 1$. If we suppose x_4 positive, the set of points on the four-sphere can be identified with the ball in three dimensions $x_1^2 + x_2^2 + x_3^2 < 1$. Similarly, supposing x_4 negative we get another copy of this ball. For x_4 zero, we must

The space of rotations is the boundary of a four-sphere with antipodal points identified.

remember the original identification of opposite points. Thus $SO(3)$ has the same structure as a sphere in four dimensions with opposite points identified. This space is called $\mathbb{R}P^3$ or real projective space. Such a space is easier to visualise in lower dimensions: $\mathbb{R}P^2$ is a sphere with opposite points identified. Such spaces come up in physics quite naturally. In liquid crystals, (look at your wristwatch or mobile to see them) there are long rod-like molecules that align to produce interesting optical properties. These molecules are long but do not point in any direction since both their ends are the same. Unlike vectors, these molecules do not have an 'arrow'. The parameter that describes the ordering alignment in liquid crystals is a headless vector, (a line segment) or a point in $\mathbb{R}P^2$.

Using the four-sphere to represent rotations is a natural trick in view of the topological ideas we went through. Such methods were used to understand tops by Cayley and Klein (the 'Cayley–Klein' parameters) long before the quantum mechanical 'spin' of elementary particles was discovered. Notice that there are two antipodal points in the four sphere which represent the same rotation. This number two is closely related to spinors. We will see that reflections also bring in this number two naturally and lead us to spinors.

3. Reflections

Euler used improper elements of $O(3)$ to understand rotations. We will write $\tilde{R}$ for reflections. Two reflections give a rotation since $\det \tilde{R}_1\tilde{R}_2 = (-1)^2 = 1$. This is easy to visualise in two dimensions. Reflecting the plane first in the line P_1 and then in the line P_2 leads to a rotation by 2ϕ. This is easy to see for vectors lying in either of the two lines P_1 or P_2. With some work you can convince yourself that it is in fact true for all vectors. You can also place two mirrors at an angle ϕ and see that the net effect of two reflections is a rotation by θ, which is twice the angle ϕ between the mirrors. Equivalently,

$$\phi = \theta/2.$$

This appearance of half angles is a characteristic of spinors! As we will see, spinors are objects that return to themselves only after a 4π rotation, unlike vectors, that return after a 2π rotation.

Place two mirrors at an angle ϕ and see that two reflections cause a rotation of 2ϕ.

In three dimensions, we reflect in a plane P whose unit normal is $\hat{p}$. The vector $\hat{p}$ is perpendicular to all vectors in P. The reflection operation is

$$\vec{r}\,' = \vec{r} - 2(\vec{r}\cdot\hat{p})\hat{p} \qquad (5)$$

and it reverses the component of $\vec{r}$ perpendicular to P, while preserving its parallel component. Note that replacing $\hat{p}$ by $-\hat{p}$ does not change matters since reflection is bilinear in $\hat{p}$. Reflecting twice in the same plane gives back the original vector

$$\vec{r}\,'' = \vec{r}\,' - 2(\vec{r}\,'\cdot\hat{p})\hat{p} = \vec{r}.$$

But reflection first in plane P_1 and then in plane P_2 gives a rotation whose axis is $\hat{n}$ and angle is θ. Evidently, the angle of rotation is twice the angle between the planes.

$$\cos(\theta/2) = \hat{p}_1\cdot\hat{p}_2.$$

The axis must lie in the intersection of planes P_1 and P_2, since this direction is unchanged in both reflections. The axis must be perpendicular to both $\hat{p}_1$ and $\hat{p}_2$ and therefore lies along $\hat{p}_1\times\hat{p}_2$ (which is non-zero unless $\hat{p}_1 \propto \hat{p}_2$).

For example, a reflection in the xy plane takes

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} x \\ y \\ -z \end{pmatrix}.$$

It simply reverses one of the coordinates of the vector (x, y, z). Instead of arranging (x, y, z) as a vector we

Rearranging the components of a vector as a matrix leads to spinors. This rearrangement was known to L Euler (1770) and O Rodrigues (1840).

could arrange it as a matrix ($X = \vec{\sigma}\cdot\vec{r} = \sigma_1 x+\sigma_2 y+\sigma_3 z$, i.e.,

$$X = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} ,$$

where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices. You can find explicit forms for these in any quantum mechanics book, but we will not need these forms. They satisfy $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1$, $\sigma_1\sigma_2 = i\ \sigma_3$, $\sigma_2\sigma_3 = i\ \sigma_1$ and $\sigma_3\sigma_1 = i\ \sigma_2$. The different σ *anticommute* with each other ($\sigma_1\sigma_2 = -\sigma_2\sigma_1$ and so on). Reflection in the xy plane is just

$$X' = -\sigma_3 X \sigma_3.$$

This follows because σ_3 *anticommutes* with σ_1, σ_2 and when the second σ_3 is moved through X to cancel the first we end up reversing only z. More generally, reflection in the plane perpendicular to $\hat{p}$ is simply

$$X' = -pXp$$

with $p = \hat{p}\cdot\vec{\sigma}$. Since $p^2 = 1$, $X'' = X$ as expected for reflections. If you reflect about P_1 followed by a *distinct* plane P_2, the result is a rotation. $X \to +p_2p_1Xp_1p_2$, where p_2p_1 is

$$p_2p_1 = (\hat{p}_2\cdot\vec{\sigma})(\hat{p}_1\cdot\vec{\sigma}) = \hat{p}_2\cdot\hat{p}_1 + i(\hat{p}_2\times\hat{p}_1)\cdot\vec{\sigma}\ .$$

The angle of rotation is given by (we can suppose $0\leq\theta\leq\pi$)

$$\cos(\theta/2) = \hat{p}_2\cdot\hat{p}_1\ .$$

The axis is given by $(\hat{p}_2\times\hat{p}_1)$. The size of the vector $\hat{p}_2\times\hat{p}_1$ is the square root of

$$\begin{aligned}(\hat{p}_2\times\hat{p}_1)\cdot(\hat{p}_2\times\hat{p}_1) &= (\hat{p}_1\cdot\hat{p}_1)\ (\hat{p}_2\cdot\hat{p}_2) - (\hat{p}_1\cdot\hat{p}_2)^2 \\ &= 1-\cos^2(\theta/2) = \sin^2(\theta/2).\end{aligned}$$

Writing $\hat{n} = (\hat{p}_2\times\hat{p}_1)/\sin(\theta/2)$ (assuming that $\theta\neq 0,\pi$) we have

$$p_2p_1 = \cos(\theta/2) + i\sin(\theta/2)\hat{n}\cdot\vec{\sigma}.$$

This useful formula can also be written as

$$p_2 p_1 = e^{i(\theta/2)\hat{n}\cdot\vec{\sigma}} \,.$$

To see this simply expand the exponential as a power series and collect terms remembering that $(\vec{\sigma}\cdot\hat{n})^{2m} = 1$. Composing rotations is easy

$$\begin{aligned}
&(\cos(\theta_2/2) + i\sin(\theta_2/2)\hat{n}_2\cdot\vec{\sigma}) \times \\
&\qquad (\cos(\theta_1/2) + i\sin(\theta_1/2)\hat{n}_1\cdot\vec{\sigma}) \\
&= \cos(\theta_2/2)\cos(\theta_1/2) - \sin(\theta_2/2)\sin(\theta_1/2)\ \hat{n}_1\cdot\hat{n}_2 \\
&+ i\cos(\theta_1/2)\sin(\theta_2/2)\ \hat{n}_2\cdot\vec{\sigma} + i\cos(\theta_2/2)\sin(\theta_1/2)\ \hat{n}_1\cdot\vec{\sigma} \\
&+ i\sin(\theta_2/2)\sin(\theta_1/2)\ (\hat{n}_1\times\hat{n}_2)\cdot\vec{\sigma}\,.
\end{aligned}$$

From this we can read off the axis and angle of the composite rotation $(\hat{n},\theta)$

$$\begin{aligned}
\cos(\theta/2) &= \cos(\theta_1/2)\ \cos(\theta_2/2) \\
&\quad - \sin(\theta_1/2)\ \sin(\theta_2/2)\ \cos(\psi)\,, \\
\sin(\theta/2)\ \hat{n} &= \cos(\theta_1/2)\ \sin(\theta_2/2)\ \hat{n}_2 \\
&\quad + \cos(\theta_2/2)\ \sin(\theta_1/2)\ \ \hat{n}_1 \\
&\quad + \sin(\theta_2/2)\ \sin(\theta_1/2)\ \hat{n}_1\times\hat{n}_2\,,
\end{aligned}$$

where we write ψ for the angle between $\hat{n}_1$ and $\hat{n}_2$.

4. Conclusion

The Pauli matrices σ^j are normally introduced as generators of rotations, satisfying *commutation* relations

$$\left[\sigma^i, \sigma^j\right] = \sigma^i\sigma^j - \sigma^j\sigma^i = 2i\epsilon^{ijk}\sigma^k,$$

where ϵ is a completely antisymmetric tensor with $\epsilon_{123} = 1$. In the present case what we used to represent reflections was the *anticommutation* relation

$$\left\{\sigma^i, \sigma^j\right\} = \sigma^i\sigma^j + \sigma^j\sigma^i = 2\ \delta^{ij}.$$

In mathematics, objects that satisfy such *anticommutation* relations are called Clifford algebras. Starting from

Pauli matrices can be viewed as generators of rotations or as elements of a Clifford algebra.

very elementary ideas, we step towards Clifford algebras and spinors.

In three dimensions we found that a 4π rotation can be deformed to the identity. This is an example of a general topological characterisation of spaces. In any space one considers different ways in which a rubber band can be placed. Let us mark a point on the rubber band and keep its location fixed. We regard two configurations of the rubber band equivalent, if one can be deformed to the other continuously. We consider two configurations and multiply them by traversing them in succession. This forms a group in the mathematical sense, called the fundamental group of the space. What we saw from the experiment is that the fundamental group of $SO(3)$ has two elements, $\{1, -1\}$. Examples of the -1 element are 2π rotations. Elements of the first are 4π, 6π or 0π.

Earlier $\vec{r}$ which was a vector in three-dimensional real space was replaced by X, a matrix in a two-dimensional complex space. Vectors in this two-dimensional complex space are called spinors. The study of elementary particles demands the use of spinors to describe particles like electrons, which have half-integral spin. Spinors form a representation of the group $SU(2)$, which can be identified with the sphere in four dimensions. This sphere 'wraps around' the space of rotations twice. We say it gives a double cover.

We learn in quantum mechanics that the wave function of a spin s object acquires a phase of $\exp(is\theta)$ when it is rotated by θ. For a spin-half particle it takes a $720°$ rotation (or a 4π rotation) for the wave function to return to itself. This is closely related to the experiment with the glass of water we started with: 4π rotations are trivial in a topological sense.

In 2 dimensions it turns out that 4π rotations cannot be shrunk to a point. Note that we essentially needed the third dimension to shrink a 4π rotation to nothing. The space of 2-d rotations is a circle and its fundamental group consists of integers, the number of times the rubber band winds around the circle. In two dimensions spin does not have to be half integral and can take any value. This has implications in two-dimensional physics, which is currently an exciting area of research.

Twp dimensions are special: spin does not have to be half integral.

In relativity we have four-vectors

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

instead of three-vectors

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} .$$

These four-vectors are easily accommodated.

Simply extend X as

$$X = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} .$$

The earlier discussion goes through. We now have spinors of the Lorentz group $SO(3,1)$ instead of the rotation group $SO(3)$. We can also write

$$X = t + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z,$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ satisfy $\mathbf{i}^2 = -1$, $\mathbf{ij} = \mathbf{k}$ and cyclic. These define the quaternions introduced by Hamilton (*Box* 2).

In closing we remark that mathematics is not a spectator sport. While it is fun to read, mathematics enters your system only if you struggle with it. To help you do this here are some exercises.

The idea of a spinor carries over to relativity theory.

Box 2. William Rowan Hamilton (1805–1865)

Hamilton invented quaternions while trying to generalise complex numbers. In Dublin, you can find a bridge (now Broom bridge, earlier Brougham bridge) on which William Rowan Hamilton carved their defining relations. Hamilton's quaternions are closely related to spinors and now very much a part of a good theoretical physics education. However at the time they were invented, their relevance to the physical world was seriously doubted. The great physicist Lord Kelvin (earlier known as William Thomsom) took a dim view of quaternions: "Quaternions came from Hamilton after his really good work had been done; and though beautifully ingenious, have been an unmixed evil to those who have touched them in any way... Vector is a useless survival, or offshoot from quaternions, and has never been of the slightest use to any creature."

Lord Kelvin may have been a great physicist, but he was a poor prophet! He did not foresee how important Hamilton's ideas would turn out to be in the microscopic world.

Exercises

1. Let R be a rotation. It can be expressed as a product of two reflections in more than one way. How many ways can this be done?

$$R = \tilde{R}_1 \tilde{R}_2 .$$

2. Given rotations R and S, we can write

$$R = \tilde{R}_1 \tilde{R}_2 ;$$

$$S = \tilde{S}_1 \tilde{S}_2 .$$

Use the freedom in 1. above to choose $\tilde{R}_2 = \tilde{S}_1$ so that these reflections cancel out. We find then

$$RS = \tilde{R}_1 \tilde{S}_2 .$$

Use this trick to derive the formula for composing rotations. Look up Hamilton's theory of turns on the web to see the connection.

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Suggested Reading

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