Invariants

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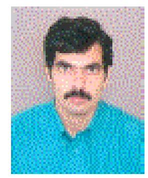
This article is on the use of 'invariants' in mathematics illustrated with some simple examples.



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In mathematics research we talk about 'problem solving' and 'theory building' research, though we cannot really compartmentalize like this. But when we are learning mathematics there is only one way – namely by solving problems, as many as we can. Of course, to find suitable problems we may have to read up some books and learn some theory! There are not too many books just on 'mathematical problems', because most interesting problems require some theoretical background, not only for solving them but also for stating them.

'The method of invariants' is a very useful technique, used in practically every field of mathematics. In his book *Problem-Solving Through Problems*, L C Loren [1] discusses the following 'techniques' of problem solving:

1. Search for a pattern.
2. Draw a figure.
3. Formulate an equivalent problem.
4. Modify the problem.
5. Choose effective notation.
6. Exploit symmetry.
7. Divide into cases.
8. Work backward.
9. Argue by contradiction.
10. Pursue parity.
11. Consider extreme cases.
12. Generalize.

As you can see 'the method of invariants' is not listed here. But essentially it is a far-reaching generalization of '10. Pursue parity'. Let us look at some examples.

Keywords
Invariants, equivalence classes, Euler characteristic.

Example 1 (Word Problem): This is taken from *Emperor's New Mind* by Roger Penrose [2]. The problem is as follows. We are given a set of equalities of words (tuples from the alphabet) as follows:

$$\begin{aligned} EAT &= AT\ , \\ ATE &= A\ , \\ LATER &= LOW\ , \\ PAN &= PILLOW\ , \\ CARP &= ME\ . \end{aligned}$$

From these we can derive $LAP = LEAP$ as follows:

$$L\underline{A}P = L\underline{AT}EP = LE\underline{ATE}P = LEAP.$$

Here underline indicates the substitutions being made. Similarly we can go to MAN from $CATERPILLAR$, as follows:

$$C\underline{ATE}RPILLAR = CARPILL\underline{A}R = CARPIL\underline{LATER}$$

$$= CAR\underline{PILLOW} = \underline{CARP}AN = ME\underline{A}N =$$

$$M\underline{EAT}EN = M\underline{ATE}N = MAN.$$

However it is not possible to go from CARPET to MEAT. How does one show this? As Roger Penrose says, such problems are part of 'non-recursive mathematics', that is, these are situations where computers cannot solve the problem (there is no algorithm) without help from us through our 'intelligence'! Before telling you how to solve this problem, let me solve for you some problems where the solution is obvious. Then we will come back to this one.

Example 2 (Zero Sum Games): Suppose three people A, B, C are playing a game. To begin with, A has Rs 50/-, B has Rs 100/- and C has Rs 200/-. After each round of the game one of the players wins and he gets 1Re each from the others. The game continues like this. (If at some stage a player has no money you can still continue the game by considering 'negative money', i.e., loan from others!). Is it possible that A, B, C have Rs 100/- each after some rounds? The answer is a clear 'No', because to begin with, the players have a total of Rs 350/- and this total remains SAME throughout! Here the 'TOTAL' does not change or it does not vary as the game progresses, so it is an *invariant.* So we cannot reach (100, 100, 100) from (50, 100, 200).

Example 3 (Difference): There are two bags with balls in them. To begin with, bag I has 25 balls and bag II has 30 balls. You toss a coin, and if it is 'Heads' you add one ball each to the bags, if it is 'Tails' you remove one ball each from the bags. By chance, if one of the bags has no ball, and you get 'Tails', simply interchange bags I and II. Here the 'difference' of balls in the bags is an invariant. So for example you cannot reach (20, 30) from (25, 30).

Both these examples have been rather trivial and one can immediately see an invariant. Here is a slightly complicated one.

Example 4 (Numerology): Start with a natural number. You are allowed to do decomposing or adding up or permuting component numbers, as described below. Say you have a number: 2458. Operation 1: Adding up some digits/components, for instance as 2+4=6, we get 658, or by considering 24+5= 29, we get 298, or as 24+58= 82, we get 82. Operation 2: Decomposing digits/components, for instance as 8=2+6, we get 24526, or as 24=23+1, we get 23158, or as 24=24+0+0, we get 240058. Operation 3: Permute digits, for instance we get 2854, 5428, etc., by permutation. Starting with a number you are allowed to do any of these operations one after another in any sequence you like. For example, you may reach 1108 from 2458, but you cannot reach 1109 from 2458. Well, just note that the remainder of the number when divided by 3 is an invariant.

We can formalize the method of invariants as follows. Suppose that we have a set S. This is the set where the 'action' is taking place. For instance in the previous example it is the set of natural numbers $\{1, 2, 3, \dots\}$. In Example 2, it is the set of triples of numbers $\{(a, b, c) : a, b, c \text{ are integers}\}$. For x, y in S, we write $x \sim y$ (read as x is *equivalent* to y), if we can go to y from x, through allowed operations. We assume the following: For all x, y, z in S,

(i) $x \sim x$; (We can stay at x if we wish to!)

(ii) $x \sim y$ implies $y \sim x$; (If we can go from x to y then we can also go from y to x.)

(iii) $x \sim y$ and $y \sim z$ then $x \sim z$; (The interpretation should be clear.)

In such a case, it is not hard to see that the set S gets partitioned into the so-called 'equivalence classes'. We say that x, y in S are in the same class if and only if $x \sim y$. Now a function f on S is an invariant if $f(x) = f(y)$, whenever $x \sim y$. For simplicity you may think of only integer-valued functions for now, but as you will see later such a restriction is not a good idea. Note that when we have an invariant function f, and you are given x, y with $f(x) \neq f(y)$, you immediately see that you cannot go to y from x. However, if $f(x) = f(y)$, in general you cannot say that it is possible to go to y from x. An invariant function f is said to be 'complete' if we can make such a statement. In other words, f is a *complete invariant*, if $f(x) = f(y)$, implies and is implied by $x \sim y$.

The set S gets partitioned into the so-called 'equivalence classes'.

Let us explain this idea through Example 4. Here we have $S = \{1, 2, 3, \dots\}$. As we noted, if we define f on S, by taking $f(x)$ as the remainder we get on dividing x by 3, then f is an invariant. However, this f is not a complete invariant. For instance, we cannot go to 10 from 7, though both give a remainder 1, when divided by 3. How do you see this? Well, we need a better invariant! If we define g on S, by taking $g(x)$, as the remainder when x is divided by 9, we see that g is an invariant and $g(7) \neq g(10)$. It is a simple exercise to show that g is a complete invariant. Usually getting a complete invariant is much harder than getting an invariant.

Now let us go back to the 'word problem' of Penrose (i.e., Example 1). We define an invariant f on 'words' by defining $f(w)$ = number of A's + number of M's + number of W's in word w. As the value of f matches on each of the given equalities, we see that f is an invariant. However, $f(CARPET) = 1$ and $f(MEAT) = 2$. Is this f a complete invariant?

The following problem is taken from the book [3] of Terence Tao who cites a book of Taylor for this problem and also gives other solutions.

Example 5 (Chameleons): In a certain island there are 13 grey, 15 brown and 17 crimson chameleons. If two chameleons of different colors meet, both of them change to the third color. No other color changes are allowed. Is it possible that after a few such color transitions all the chameleons have the same color?

You can see that if a, b and c denote the number of chameleons of grey, brown and crimson colors respectively, then under each transition the new members are given by adding $(2, -1, -1)$, or $(-1, 2, -1)$, or $(-1, -1, 2)$ to (a, b, c). The sum $a + b + c$ is an invariant but is of no use for the problem we have. It is seen that a weighted sum such as $f((a, b, c)) = 0.a + 1.b + 2.c$ is an invariant under modulo 3 addition, and it does the job. Is it a complete invariant? So far all the examples have been artificial or 'cooked up' problems. Here are two well-known puzzles, which just need 'pursuing parity' to solve them.

Example 6 (Chess board): Consider a 8×8 square with two opposite corner squares missing.

So now we have $64 - 2 = 62$ squares. The question is whether we can cover this with thirty one pieces of 2×1 rectangles. The answer is No. Just suppose that you have started with a normal chess board so that the each cell is colored black or white. Note that you have removed two cells of the same color. Whenever you are putting a 2×1 rectangle you are covering one black and one white cell. So the total number of white cells you have covered minus the total number of black cells you have covered after putting some 2×1 rectangles is always zero. So this difference is an invariant! You cannot make it reach ± 2.

Example 7 (Fifteen Puzzle): This is another well-known puzzle. I will not explain the set-up here. If you have not seen it before, refer to the web pages [4], or see the *Resonance* article [5]. 'From which permutations of $\{1, 2, ...15\}$ can we reach the natural order (identity permutation)?' is the question. Well, the set of permutations are divided into two classes, odd and even permutations. Odd permutations are the ones obtained by an odd number of transpositions (exchanges between exactly two positions). I leave it you to figure out that in this puzzle, every movement keeps the 'parity', that is, from an odd permutation you can only go to an odd permutation and from an even permutation you can only go to an even permutation. In other words this 'parity' is an invariant here. Show that it is in fact a complete invariant.

Example 8 (ISBN): For most published books you find a code number called ISBN number. For example the book of Penrose quoted above has ISBN number: 0-09-977170-5. Given any such ISBN number say, $d_1 d_2 d_3 ... d_9 d_{10}$, we see that the remainder of $d_1 + 2d_2 + 3d_3 + \cdots + 9d_9$, when divided by 11 is d_{10}. (What is the invariant here?). ISBN code numbers are given this way just to detect errors. The invariant here can always detect single digit errors. If you see the ISBN number of Larson's book you will have a small surprise! See the recent articles [6] for more on ISBN and barcodes.

Now let us study some invariants which have actually appeared in mathematics. Perhaps the best known is the 'Euler characteristic'. I will explain this only through pictures without proper definitions, and without being very precise.

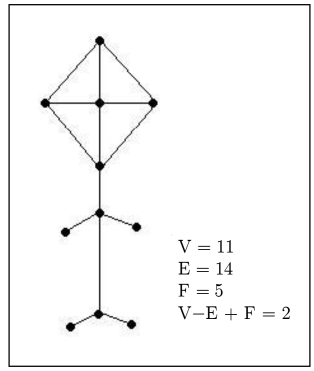

Figure 1.

Example 9 (Euler Characteristic): Consider a picture (connected planar graph) as shown in *Figure* 1. If we take V = number of vertices, E = number of edges F = number of faces (or regions, including 'outside'), then $V - E + F = 2$ always . In other words, $V - E + F$ is an invariant. It does not change by deleting some edges, or by deleting some vertices and edges attached to them, or by adding more vertices and edges. Here $V - E + F$ is an invariant, known as the 'Euler characteristic'.

So far all the invariants we have considered are integers. But it is clearly not necessary that it should be an integer. It can be any real number, or it could be any complex number. It need not even be a number. It could be a polynomial, or in other words it could be a collection of numbers. Actually, it could be any sort of mathematical object whatsoever. Mathematicians have stretched this to unimaginable levels. For example to classify surfaces (or topological spaces) the invariants could be numbers or groups or algebras. Or to classify algebras you may use groups as invariants as in K-theory of C^*-algebras. The main thing is that it should be possible to compute invariants at least in some cases and they should help us to distinguish classes.

Here $V - E + F$ is an invariant, known as the 'Euler characteristic'.

In fact, Banach and Tarski showed that the unit sphere can be broken up into finitely many pieces and re-arranged to have two unit spheres!

Example 10 (Discriminant): Consider a quadratic polynomial $p(x) = ax^2 + bx + c$. Now if we evaluate this polynomial at a shifted point, say at $x + d$, we get a different polynomial, namely $q(x) = p(x + d) = a(x + d)^2 + b(x + d) + c = Ax^2 + Bx + C$, where $A = a, B = 2ad + b, C = ad^2 + bd + c$. We see that the discriminant is an invariant here, that is, $b^2 - 4ac = B^2 - 4AC$. This shows that the number of real roots of the polynomials will be the same, etc.

Example 11 (Area, Volume): Suppose we have the unit square $[0, 1] \times [0, 1]$. Can we cut it into some pieces and re-arrange to have the square $[0, 2] \times [0, 1]$. We say that this is not possible, because the area is an invariant of this process. However, this proof does not work, if the pieces have no well-defined area. In fact, Banach and Tarski showed that the unit sphere can be broken up into finitely many pieces and re-arranged to have two unit spheres! Of course, these pieces are quite irregular and have no well-defined volume. This is known as Banach–Tarski paradox [7, 8].

V Jones found an invariant called 'Jones index' for some pairs of algebras known as subfactors which takes precisely the values

$$\{4\cos^2\frac{\pi}{n}, n \geq 3, n \in \mathbb{N}\} \bigcup [4, \infty].$$

He also found an invariant for knots (or links) known as Jones polynomial. This earned him the Fields medal.

Finally here is a problem where we do not know of suitable invariants.

Example 12 ($3n + 1$ Problem) : Start with a natural number n, bigger than 1. If it is even divide by 2, that is, move to $\frac{n}{2}$, if it is odd then multiply by 3 and add 1, that is, move to $3n + 1$. Continue this procedure. The following question is open: Whether starting with any natural number n bigger than 1, you finally come down to 1. Nobody knows the answer! For more on this problem, see [9] or [10].

Suggested Reading

[1] Loren C Larson, *Problem-Solving Through Problems*, Springer-Verlag (1983).

[2] Roger Penrose, *The Emperor's New Mind*, Vintage Edition, Oxford University Press, 1989.

[3] Terence Tao, *Solving Mathematical Problems, a Personal Perspective*, Oxford University Press, 2006.

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