Plane Curves of Some Geometrical Significance and Associated Differential Equations

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Loci of points in the Euclidean plane are determined from prescribed relations of the points with given points, and/or, lines. The dependence of these relations on parameters lead to the differential equations representing the family of loci under concern. Incidentally most of the differential equations thus obtained are nonlinear ordinary differential equations.

1. Introduction

The fundamental concepts in a Euclidean plane are 'points' and 'lines'. In this article we shall explore how prescribed relations with given points and lines lead to certain loci in the Euclidean plane, and to certain differential equations. It is not surprising that these loci consist of well-known conics on a Euclidean plane or parts of them. By eliminating the parameters present in the prescribed relation, the differential equation representing the family of plane curves under concern is obtained. It is found that these differential equations are nonlinear unless the distances of the points on the loci from two given lines are considered. Section 2 deals with geometrical relations involving distances of the points on the locus required from one given point, two given points or one given point and one given line. In Section 3, geometrical relations involving distances of the points on the required locus from two given lines are considered.

The differential equations are generally not linear.

2. Loci of Points Satisfying Prescribed Geometrical Relations with One Given Point/Two Given Points/One Given Point and One Given Line

The distance between two given points P, Q in a Euclidean plane is denoted by $d(P,Q)$, and the distance of a point P from a line l in a Euclidean plane is denoted by

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Euclidean plane, distance, ratio, sum, difference, differential equations.

The required locus in the case given in **2.2** is either a circle or a straight line.

Let $k \geq 0$ be a prescribed number.

2.1 Given one point S and the prescribed geometrical relation (A): $d(P,S) = k$, the equation of the required locus of the points $P(x,y)$ in the Euclidean plane (with $S \equiv (0,0)$) that satisfy (A), is found to be

$$x^2 + y^2 = k^2. \quad (1)$$

The associated differential equation is obtained by eliminating k between (1) and the equation derived from (1) on differentiating with respect to x $(' \equiv \frac{\mathrm{d}}{\mathrm{d}x})$.

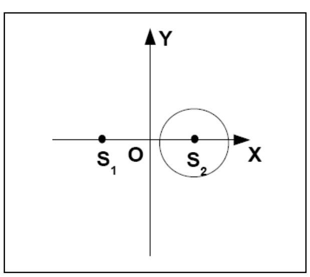
$$x + yy' = 0. \quad (2)$$

2.2 Given two points S_1, S_2 and the prescribed geometrical relation (A): $d(P,S_1) = kd(P,S_2)$, the equation of the required locus of the points $P(x,y)$ in the Euclidean plane (with $S_1 \equiv (-a,0)$, $S_2 \equiv (a,0)$), that satisfy (A), turns out to be

$$(1-k^2)(x^2+y^2) + 2a(1+k^2)x + a^2(1-k^2) = 0. \quad (3)$$

Hence, the required locus is a circle if $k \neq 1$; it is a straight line if $k = 1$ (which is actually the orthogonal bisector of the line segment S_1S_2). If $k > 1$, the point S_2 lies inside the circle given by (3), while the points S_1 and $O \equiv (0,0)$ lie outside it (*Figure* 1). If $k < 1$, the point S_1 lies inside the circle given by (3), while the points S_2, O lie outside it. In both the cases, the points S_1, S_2 are such that the polar of one with respect to the circle (3) passes through the other.

Figure 1.



The associated differential equation is obtained by eliminating k between (3) and the equation derived from (3) on differentiating with respect to x.

$$y^2 - x^2 - 2xyy' + a^2 = 0, \quad (4)$$

if S_1, S_2 are kept fixed.

If S_1, S_2 are also allowed to vary, the associated differential equation becomes

$$1 + y'^2 + y'' = 0, \qquad (5)$$

which is obtained by eliminating a between (4) and the equation derived from (4) on differentiating with respect to x.

The associated differential equation is non-linear.

2.3 Given two points S_1, S_2 and the prescribed geometrical relation as (A): $d(P, S_1) + d(P, S_2) = k$ or (B): $|d(P, S_1) - d(P, S_2)| = k$, the equation of the required locus of points $P(x, y)$ in the Euclidean plane (with $S_1 \equiv (-a, 0)$, $S_2 \equiv (a, 0)$), that satisfy relation (A) or (B) turns out to be

$$4(k^2 - 4a^2)x^2 + 4k^2y^2 = k^2(k^2 - 4a^2)\,. \qquad (6)$$

It is to be noted that

$$k = d(P, S_1) + d(P, S_2) \geq d(S_1, S_2) = 2a,$$

while

$$k = |d(P, S_1) - d(P, S_2)| \quad d(S_1, S_2) = 2a.$$

If $k > 2a$, (6) represents an ellipse (*Figure* 2), while for $k < 2a$, (6) represents a hyperbola (*Figure* 3). For $k = 2a$, (6) reduces to $y = 0$. In this case, the geometrical relation (A) determines the line segment S_1S_2 as the required locus (*Figure* 4), and for the geometrical relation (B), the required locus turns out to be the union of the rays $\{y = 0, x \geq a\}$, $\{y = 0; x \quad - a\}$ (*Figure* 5).

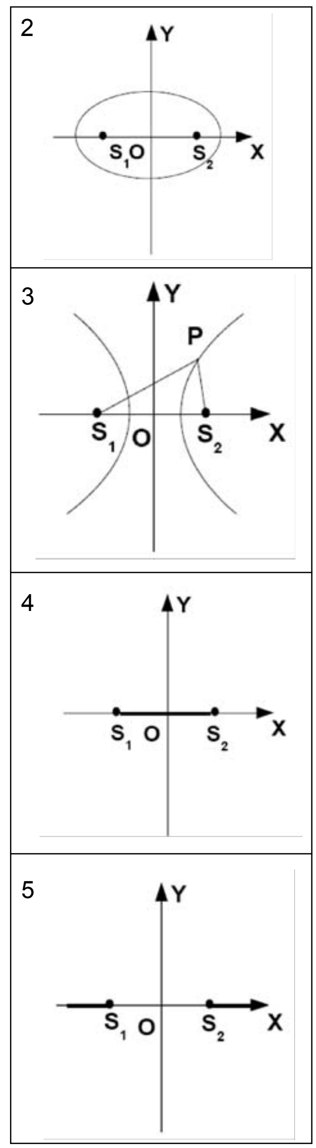
If the given points S_1, S_2 are kept fixed, the associated differential equation is found to be

$$xyy'^2 + (x^2 - y^2 - a^2)y' - xy = 0, \qquad (7)$$

while, if the given points S_1, S_2 are allowed to vary, the associated differential equation turns out to be

$$x(y'^2 + yy'') - yy' = 0. \qquad (8)$$

Figures 2–5.



The locus in the case of **2.3** is either an ellipse or a hyperbola. In the degenerate case it is either a line segment or a pair of rays.

Equations (7) and (8) are obtained by following the same procedure as described in Section 2.2 for deriving (4) and (5) respectively.

2.4 Given a point S and a line l, with the prescribed geometrical relation (A): $d(P,S) = kd(P;l)$, the required locus of the points $P(x,y)$ in the Euclidean plane (with $S \equiv (a,0)$ and $l : x = 0$), that satisfy (A), turns out to be

$$(1-k^2)x^2 - 2ax + y^2 + a^2 = 0. \qquad (9)$$

Thus the required locus is an ellipse if $k < 1$, a hyperbola if $k > 1$, and a parabola if $k = 1$. It can be shown that, in each case, the given point S is a focus of the locus sought for, and the prescribed ratio k is its eccentricity.

If the distance between the given point S and the given line l is kept fixed, the associated differential equation becomes

$$xyy' = y^2 + a(a-x). \qquad (10)$$

When the distance between the given point S and the given line l varies, the associated differential equation is

$$xyy' - y^2 = \{yy' - x(yy')'\}\{yy' - x(yy')' - x\}. \qquad (11)$$

2.5 Given a point S and a line l, with the prescribed geometrical relation (A): $d(P,S) + d(P;l) = k$, the required locus of the points $P(x,y)$ in the Euclidean plane (with $S \equiv (0,0)$ and $l : x = a$), that satisfy (A), is

$$y^2 = -2(a \mp k)x + (a \mp k)^2$$

according as $x \quad a$ or $x \geq a$; that is,

$$y^2 = 2(k-a)x + (k-a)^2 \quad \textit{if } x \quad a, \qquad (12)$$

$$y^2 = -2(k+a)x + (k+a)^2 \quad \textit{if } x \geq a. \qquad (13)$$

The locus in the case of **2.4** is either an ellipse, a hyperbola or a parabola.

It can be shown that the focus of each of the parabolas (12), (13) is at the given point S.

For $k > a$, the required locus turns out to be a closed curve, composed of the portion of the parabola (12) with $x \quad a$, and the portion of the parabola (2.13) with $x \geq a$ (*Figure* 6).

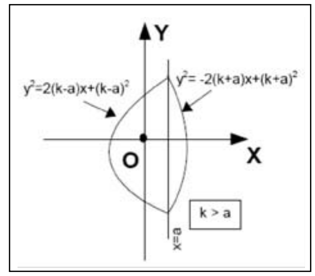


Figure 6.

For $k < a$, it follows from (12), $x \leq \frac{a-k}{2} < a$;
then $k = d(P, S) + d(P; l)$
$= \sqrt{x^2 + 2(k-a)x + (k-a)^2} + a - x = a - k - x + a - x$, so that $x = a - k$ which contradicts $x \leq \frac{a-k}{2} < a$. For $x \geq a$, it follows from (13) that $x \leq \frac{a+k}{2} < a$, which contradicts $x \geq a$. Hence, for $k < a$, the required locus is empty.

For $k = a$, (12) reduces to $y = 0$ $(x \leq a)$. For $x < 0$ if $P' \equiv (x, 0)$ then

$$k = d(P', S) + d(P'; l) = |x| + a - x > a = k,$$

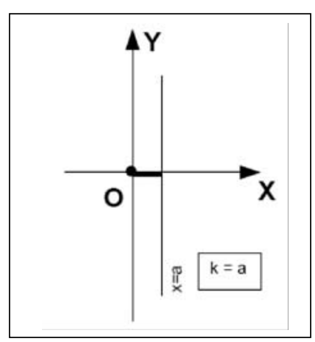
which is impossible, so the required locus contains no point $P'(x, 0)$ with $x < 0$. And, for $k = a$, (13) reduces to $y^2 = -4a(x - a)$, $(x \geq a)$; so the required locus contains no point $P''(x, 0)$ with $x \geq a$. Hence, the required locus for $k = a$, is the line segment $y = 0$, $0 \quad x \leq a$ (*Figure* 7).

2.6 Given a point S and a line l, with the prescribed geometrical relation (A): $|d(P, S) - d(P; l)| = k$, the required locus of the points $P(x, y)$ in the Euclidean plane (with $S \equiv (0, 0)$ and $l : x = a$) is to be considered separately for

(I) $d(P, S) \geq d(P; l)$

(II) $d(P, S) \quad d(P; l)$.

Figure 7.

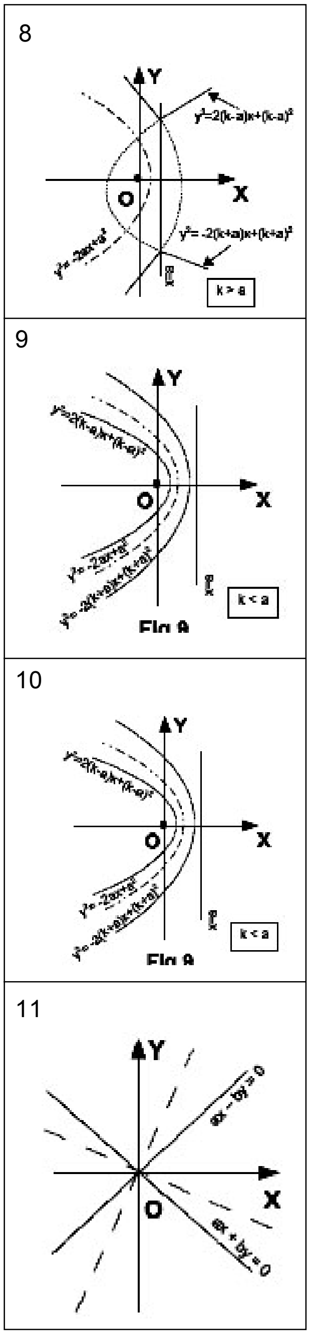


In case (I), the required locus consists of the portion of the parabola

$$y^2 = 2(k-a)x + (k-a)^2 \quad \textit{for} \quad x \geq a, \qquad (14)$$

and the portion of the parabola

$$y^2 = -2(k+a)x + (k+a)^2 \quad \textit{for} \quad x \leq a. \qquad (15)$$



Figures 8–11.

In case (II), the points $P(x, y)$ on the required locus satisfy

$$y^2 = 2(k-a)x + (k-a)^2 \quad \textit{for} \;\; x \geq a. \tag{16}$$

It turns out that:

For $k > a$, the required locus consists of (i) the portion of the parabola (14) for which $x \geq a$, plus its portion with $d(P, S) \leq d(P; l)$, and (ii) the portion of the parabola (15) with $x \leq a$ (*Figure* 8).

For $k < a$, the required locus consists of the two parabolas (14) and (15) (*Figure* 9).

For $k = a$, the required locus consists of (i) the parabola $y^2 = -4ax + 4a^2$ and (ii) the line $y = 0$ with $x \leq \frac{a}{2}$, $x \geq a$ (*Figure* 10).

It can be shown that the differential equation having the parabolas given by (12) (same as (14)) and (13) (same as (15)) as its solution-curves is

$$yy'^2 + 2xy' - y = 0. \tag{17}$$

3. Loci of Points Satisfying Prescribed Geometrical Relations with Two Given Lines

3.1 Given two intersecting lines l_1, l_2, and the prescribed geometrical relation (A): $d(P; l_1) = kd(P; l_2)$, the equation of the required locus of the points $P(x, y)$ (with $l_1 : ax + by = 0, \;\; l_2 : ax - by = 0, \;\; a \geq 0, \;\; b \geq 0$), that satisfy (A) is (*Figure* 11)

$$ax + by = \pm k(ax - by). \tag{18}$$

3.2 Given two intersecting lines l_1, l_2, and the prescribed geometrical relation (A): $d(P; l_1) + d(P; l_2) = k$, (B): $|d(P; l_1) - d(P; l_2)| = k$, the equation of the required locus of the points $P(x, y)$ in the Euclidean plane (with $l_1 : ax + by = 0,\ \ l_2 : ax - by = 0,\ \ a \geq 0,\ \ b \geq 0$) that satisfy (A) is

$$|ax + by| + |ax - by| = k\sqrt{(a^2 + b^2)}, \tag{19}$$

and that satisfy (B) is

$$||ax + by| - |ax - by|| = k\sqrt{(a^2 + b^2)}. \tag{20}$$

In case of geometrical relation (A), the required locus turns out to be a rectangle if $a \neq b$ (*Figure* 12), and a square if $a = b$. In case of geometrical relation (B), the required locus is found to consist of the four rays (*Figure* 13):

$$\begin{aligned} x = \pm k\sqrt{(a^2+b^2)}/2a,\ y \leq &\ -\frac{k}{2b}\sqrt{(a^2+b^2)}, \\ y \geq &\ \frac{k}{2b}\sqrt{(a^2+b^2)}; \\ y = \pm k\sqrt{(a^2+b^2)}/2b,\ x \leq &\ -\frac{k}{2a}\sqrt{(a^2+b^2)}, \\ x \geq &\ \frac{k}{2a}\sqrt{(a^2+b^2)}. \end{aligned} \tag{21}$$

3.3 Given two parallel lines l_1, l_2, and the prescribed geometrical relation (A): $d(P; l_1) = kd(P; l_2)$, the equation of the required locus of the points $P(x, y)$ in the Euclidean plane (with $l_1 : x = -a$, $l_2 : x = a$, $a > 0$), that satisfy (A) is found to be,

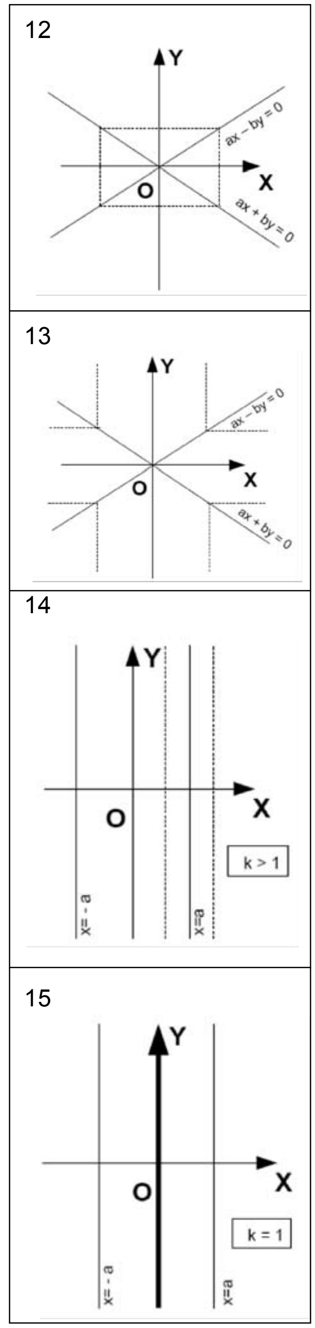
$$\begin{aligned} for\ k \neq 1,\ (1-k)x = (1+k)a\ if\ |x| \geq a, \\ (1+k)x = (1-k)a\ if\ |x| \leq a, \end{aligned} \tag{22}$$

(*Figure* 14).

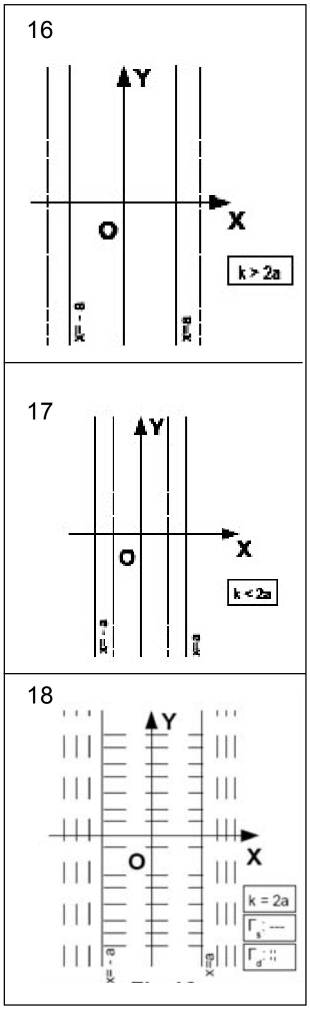
$$and\ for\ k = 1, \qquad x = 0. \tag{23}$$

(*Figure* 15).

3.4 Given two parallel lines l_1, l_2, and the prescribed geometrical relation (A): $d(P; l_1) + d(P; l_2) = k$, (B):



Figures 12–15.



Figures 16-18.

$|d(P;l_1) - d(P;l_2)| = k$, the equation of the required locus of the points $P(x,y)$ in the Euclidean plane (with $l_1 : x = -a$, $l_2 : x = a$; $a > 0$) is found to be

$$|x-a| + |x+a| = k \tag{24}$$

for points satisfying (A), and

$$||x-a| - |x+a|| = k \tag{25}$$

for points satisfying (B).

In the cases $k > 2a$, $k < 2a$, the required locus is a pair of lines, parallel to the given parallel lines (*Figures* 16 and 17).

For $k = 2a$, the set of points, the sum of whose distances from the two given parallel lines equals the distance between the given parallel lines, forms the strip $-a \le x \le a$, and the set of points, the difference of whose distances from the two given parallel lines equals the distance between the two given parallel lines, is the union of the two half planes $x \le -a$ and $x \ge a$ (*Figure* 18) .

4. Conclusion

In the above, the loci of points in the Euclidean plane, satisfying prescribed geometrical relations, have been found. Also found are the differential equations having the family of loci under concern as the solution-curves of the corre- sponding differential equation. In most of the cases, the associated differential equation is nonlinear. Solving nonlinear differential equations is not easy in general, and there is no known method of determining geometrical properties of the solution-curves of nonlinear differential equations. As the differential equations, have been derived here starting from prescribed geometrical relations, at least for these nonlinear differential equations, it has been possible to highlight the geometrical properties of their solution-curves.

Solving nonlinear differential equations is not easy in general.