Motion of a Tiny Tool Thrown by an Astronaut towards another Astronaut inside a Spinning Space Vehicle in a State of Free Fall Revisited.

S N Maitra
Mathematics Department
National Defence Academy
Khadakwasla
Pune 411 023, India.
Email:
soumen_maitra@yahoo.co.in

A space vehicle rotating with a uniform angular velocity about a vertical axis fixed to it is falling freely vertically downwards, say, with its engine shut off. It carries two astronauts inside it. One astronaut throws a tiny tool towards the other astronaut. The motion of the tiny tool with reference to a rotating frame rigidly fixed to the space vehicle, i.e., relative to the astronauts, is considered. Second-order differential equations of its motion instead of fourth order [1] are obtained and are completely solved with exposure of a subtle method of solution, from pedagogic viewpoint utilizing the relevant initial conditions of its projection. Under certain conditions of projection of the tiny tool, it is established that it begins to float in the space inside the vehicle after describing an analytical maximum distance. The position and velocity of the tiny tool at any instant of time with reference to an inertial frame defined by the non-rotating frame initially coincident with the rotating frame at the time of projection of the tiny tool, are also found out. Finally two numerical examples are given.

1. Introduction

Das and his coauthors [1] determined the trajectory of a tiny tool thrown by one astronaut towards another astronaut inside a space vehicle rotating about an axis fixed to it with a uniform, angular velocity, being in the state of free fall. They considered its motion with reference to the rotating frame, i.e., as viewed by the astronauts. Because of free-fall of the vehicle and the tiny

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tool, both are acted on by the gravitational force in the same vertical direction entailing no effect of gravity as regards the motion of the tiny tool as a particle relative to the astronauts. They have considered a system of axes OXYZ ridigly fixed to the space vehicle with origin O; the axes OX, OY and OZ are mutually perpendicular to each other. Hence this is a rotating frame with angular velocity ω equal to that of the space vehicle about OX, i.e., the Z-axis.

By the theorem [1,2] of composition of velocity and acceleration, the acceleration vector of the tiny tool with respect to the rotating frame is given by

$$\vec{f}_r = \frac{\mathrm{d}\vec{v}_r}{\mathrm{d}t} = (\vec{\omega}\cdot\vec{\omega})\vec{r} + 2\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)\times\vec{\omega}\ , \qquad (1)$$

where $\vec{r}$ is the position vector and $\vec{v}_r$ the velocity of the tiny tool after a time t of throwing it. Das and his coauthors rewrote (1) in scalar form as

$$\frac{\mathrm{d}^2 x_r}{\mathrm{d}t^2} = 2\omega\frac{\mathrm{d}y_r}{\mathrm{d}t} + \omega^2 x_r\ , \qquad (2)$$

$$\frac{\mathrm{d}^2 y_r}{\mathrm{d}t^2} = -2\omega\frac{\mathrm{d}x_r}{\mathrm{d}t} + \omega^2 y_r\ , \qquad (3)$$

obviously taking

$$\vec{\omega} = \hat{e}_{0,z}\omega,\ \vec{r} = \hat{e}_{0,x}x_r + \hat{e}_{0,y}y_r\ , \qquad (4)$$

$$\vec{v}_r = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \hat{e}_{0,x}v_{r,x} + \hat{e}_{0,y}v_{r,y}\ , \qquad (5)$$

where (x_r, y_r) is the position of the tiny tool at time t and $\hat{e}_{0,x}, \hat{e}_{0,y}, \hat{e}_{0,z}$ represent unit vectors along OX, OY, OZ respectively. Without describing the elimination technique, using (2) and (3) they [1] wrote two fourth-order differential equations

$$\frac{\mathrm{d}^4 x_r}{\mathrm{d}t^4} + 2\omega^2\frac{\mathrm{d}^2 x_r}{\mathrm{d}t^2} + \omega^4 x_r = 0, \quad \frac{\mathrm{d}^4 y_r}{\mathrm{d}t^4} + 2\omega^2\frac{\mathrm{d}^2 y_r}{\mathrm{d}t^2} + \omega^4 y_r = 0\,, \qquad (6)$$

and then wrote their solutions as

$$x_r = (v_{0,x} + R\omega)t\cos\omega t + (v_{0,y}t - R)\sin\omega t\,,$$
$$y_r = (tv_{0,y} - R)\cos\omega t - (v_{0,x} + R\omega)t\sin\omega t\,, \quad (7)$$

assuming the initial condition that the tool is thrown from a point $(0, -R, 0)$, i.e., lying on the negative side of the Y-axis of the rotating frame, with a velocity whose components along OX and OY are $v_{0,x}$ and $v_{0,y}$ respectively. That is, the initial conditions of projection are

$$x_r = 0, \quad y_r = -R, \quad z_r = 0, \quad \text{at } t = 0, \qquad (8)$$

$$\vec{v}_r(t=0) = \hat{e}_{0,x}v_{0,x} + \hat{e}_{0,y}v_{0,y}\,. \qquad (9)$$

They [2] also utilized further initial conditions

$$\left.\frac{\mathrm{d}^3x_r}{\mathrm{d}t^3}\right|_{t=0} = 0; \quad \left.\frac{\mathrm{d}^3y_r}{\mathrm{d}t^3}\right|_{t=0} = 0\,. \qquad (10)$$

2. Alternative Method of Solutions

Herein are solved the differential equations in a simpler way involving complex number $i(=\sqrt{-1})$ with exposure of the method of solution. The complete trajectory of the tiny tool is obtained after throwing it from a point $(a, b, 0)$ lying on the XY-plane of the rotating frame with the same velocity components $(v_{0,x}, v_{0,y})$.

Multiplying (3) by i and then respectively adding to and subtracting from (2), one gets on rearrangement:

$$\frac{\mathrm{d}^2Z}{\mathrm{d}t^2} + 2i\omega\frac{\mathrm{d}Z}{\mathrm{d}t} - \omega^2 Z = 0\,, \qquad (11)$$

$$\frac{\mathrm{d}^2\bar{Z}}{\mathrm{d}t^2} - 2i\omega\frac{\mathrm{d}\bar{Z}}{\mathrm{d}t} - \omega^2 \bar{Z} = 0\,, \qquad (12)$$

where

$$Z = x_r + iy_r \quad \text{and} \quad \bar{Z} = x_r - iy_r\,. \qquad (13)$$

The complementary functions of (11) are due to the equations

$$(D^2+2\omega iD-\omega^2)Z=0 \quad \text{or } (D^2+2\omega iD+(i\omega)^2)Z=0,$$

i.e., $(D+\omega i)^2=0$ having two equal roots $(-\omega i)$ so that

$$Z=x_r+iy_r=(A_1+B_1t)e^{-i\omega t}\ . \tag{14}$$

Similarly the complementary functions of (12) are:

$[D^2-2i\omega d+(i\omega)^2]\bar{Z}=0$ having two equal roots (ωi):

$$\bar{Z}=x_r-iy_r=(A_2+B_2t)e^{i\omega t}\ , \tag{15}$$

where A_1,B_1,A_2,B_2 are constants to be evaluated from the intial conditions (8) and (9).

Combining (13), (14) and (15) in two different manners and using the formulae $e^{\pm i\omega t}=\cos\omega t\pm i\sin\omega t$:

$$\begin{aligned} x_r &= \left(\frac{A_1+A_2}{2}+\frac{B_2+B_1}{2}t\right)\cos\omega t \\ &+ i\left(\frac{A_2-A_1}{2}+\frac{B_2-B_1}{2}t\right)\sin\omega t\ , \end{aligned} \tag{16}$$

$$\begin{aligned} y_r &= i\left(\frac{A_2-A_1}{2}+\frac{B_2-B_1}{2}t\right)\cos\omega t \\ &- \left(\frac{A_2+A_1}{2}+\frac{B_2+B_1}{2}t\right)\sin\omega t\ . \end{aligned} \tag{17}$$

These can be rewritten involving four different constants A,B,C,D as

$$x_r=(A+Bt)\cos\omega t+(C+Dt)\sin\omega t\ , \tag{18}$$

$$y_r=(C+Dt)\cos\omega t-(A+Bt)\sin\omega t\ . \tag{19}$$

3. Trajectory of the Tiny Tool in the Space Vehicle

In consequence of (18) and (19) and applying the initial conditions $t = 0$, $x_r = a$ and $y_r = b$ that yield

$$A = a \quad \text{and} \quad C = b\,, \tag{20}$$

the position of the tiny tool as observed by the astronauts is

$$x_r = (a + Bt)\cos\omega t + (b + Dt)\sin\omega t\ , \tag{21}$$

$$y_r = (b + Dt)\cos\omega t - (a + Bt)\sin\omega t\ . \tag{22}$$

Differentiating (21) and (22) with respect to time t we get the components of the velocity with reference to the rotating frame OXYZ, i.e., as viewed by the astronauts:

$$\begin{aligned} v_x = \frac{\mathrm{d}x_r}{\mathrm{d}t} &= -\omega(a + Bt)\sin\omega t + B\cos\omega t \\ &+ \omega(b + Dt)\cos\omega t + D\sin\omega t\ , \end{aligned} \tag{23}$$

$$\begin{aligned} v_y = \frac{\mathrm{d}y_r}{\mathrm{d}t} &= -\omega(b + Dt)\sin\omega t + D\cos\omega t \\ &- \omega(a + Bt)\cos\omega t - B\sin\omega t\,. \end{aligned} \tag{24}$$

By using the initial conditions of projection

$$t = 0, \quad \frac{\mathrm{d}x_r}{\mathrm{d}t} = v_{0,x} \quad \text{and} \quad \frac{\mathrm{d}y_r}{\mathrm{d}t} = v_{0,y}\ , \tag{25}$$

we get

$$B = v_{0,x} - \omega b, \quad D = v_{0,y} + \omega a\ . \tag{26}$$

Substituting (26) in (21) and (22), we get

$$x_r = \{a + (v_{0,x} - \omega b)t\}\cos\omega t + \{b + (v_{0,y} + \omega a)t\}\sin\omega t\,, \tag{27}$$

$$y_r = \{b + t(v_{0,y} + \omega a)t\}\cos\omega t - \{a + (v_{0,x} - \omega b)t\}\sin\omega t\,, \tag{28}$$

which give the trajectory of the tiny tool with reference to the rotating frame, i.e., relative to the astronauts.

Using (26) in (23) and (24) or differentiating (27) and (28) we get the velocity components:

$$\begin{aligned} v_x &= \{v_{0,y} - \omega(v_{0,x} - \omega b)t\}\sin\omega t \\ &+ \{v_{0,x} + \omega(v_{0,y} + \omega a)t\}\cos\omega t\,, \end{aligned} \tag{29}$$

$$\begin{aligned} v_y &= -\{v_{0,x} + \omega(v_{0,y} + \omega a)t\}\sin\omega t \\ &+ \{v_{0,y} - \omega(v_{0,x} - \omega b)t\}\cos\omega t\,. \end{aligned} \tag{30}$$

Squaring and adding (29) and (30), we get the velocity v_r:

$$v_r^2 = [v_{0,x} + \omega(v_{0,y} + \omega a)t]^2 + [v_{0,y} - \omega(v_{0,x} - \omega b)t]^2\,. \tag{31}$$

4. Floatation of the Tiny Tool

Equation (31) suggests that the velocity of the tiny tool relative to the astronauts, i.e., as observed by them, vanishes after a time t:

$$t = \frac{-v_{0,x}}{\omega(v_{0,y} + \omega a)} = \frac{v_{0,y}}{\omega(v_{0,x} - \omega b)}, \tag{32}$$

if the following relationship as regards the initial conditions of projection is fulfilled with the initial velocity v_0 of projection (by cross multiplication of (32))

$$v_0^2 = v_{0,x}^2 + v_{0,y}^2 = \omega(bv_{0,x} - av_{0,y}). \tag{33}$$

Hence (32) can be rewritten as

$$t = \frac{-(bv_{0,x} - av_{0,y})^2}{(bv_{0,y} + v_{0,x}a)v_0^2} > 0 \tag{34}$$

which necessitates

$$bv_{0,y} + v_{0,x}a < 0 \tag{35}$$

depending on the position/direction of projection of the tiny tool. Hence it begins to float after a time t given by (34) subject to the conditions (33), (34) and (35). The point at which it floats is given by (x'_r, y'_r) by substituting for t from (34) in (27) and (28). Needless to mention that the foregoing floatation of the tiny tool is observed by the astronauts.

5. Discussion and Conclusion

The velocity and displacement expressions, (27) to (29), can also be expressed as

$$v_x = \dot{x}_r = A_1 \cos(\omega t - \varepsilon_1), \quad v_y = \dot{y}_r = -A_1 \sin(\omega t - \varepsilon_1)\,, \tag{36}$$

$$x_r = A_2 \cos(\omega t - \varepsilon_2), \quad y_r = -A_2 \sin(\omega t - \varepsilon_2)\,, \tag{37}$$

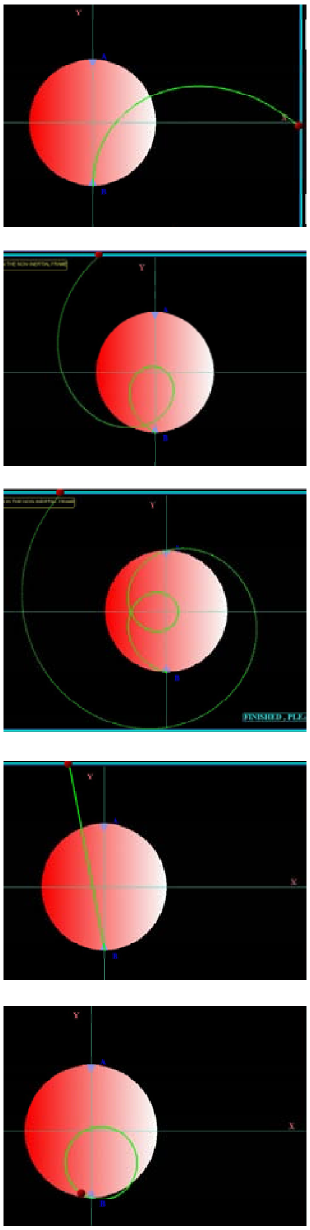
where

$$\begin{aligned} A_1 &= \sqrt{[v_{0,x} + \omega(v_{0,y} + \omega a)t]^2 + [v_{0,y} - \omega(v_{0,x} - \omega b)t]^2} = v_r \\ A_2 &= \sqrt{a + (v_{0,x} - \omega b)t]^2 + [b + (v_{0,y} + \omega a)t]^2} = r\,. \end{aligned} \tag{38}$$

$$\begin{aligned} \tan\varepsilon_1 &= \frac{v_{0,y} - \omega(v_{0,x} - \omega b)}{v_{0,x} + \omega(v_{0,y} + \omega a)} \quad , \\ \tan\varepsilon_2 &= \frac{b + (v_{0,y} + \omega a)t}{a + (v_{0,x} - \omega b)t}\,. \end{aligned} \tag{39}$$

Differentiating (31) with respect to t and equating to zero, gives

$$t = -\frac{av_{0,x} + bv_{0,y}}{(v_{0,y} + \omega a)^2 - (v_{0,x} - \omega b)^2} > 0 \tag{40}$$



Figures (1–5) reproduced from reference [1].

which is the time taken by the tool to attain the maximum velocity. Hence the maximum distance travelled by it can be obtained by putting the value of t in A_2 from (34) and the maximum velocity in time t is due to (38) and (40).

The velocity vector $\vec{v}$ with respect to an inertial frame defined earlier is given by

$$\begin{aligned}\vec{v} &= v_{0,x}\hat{e}_{0,x} + v_{0,y}\hat{e}_{0y} + \omega\hat{e}_{0,z} \times (a\hat{e}_{0,x} + b\hat{e}_{0,y}) \\ &= (v_{0,x} - b\omega)\hat{e}_{0,x} + (v_{0,y} + a\omega)\hat{e}_{0,y} \end{aligned} \quad (41)$$

and the trajectory of the tiny tool is given by (in the same inertial frame)

$$x_r = a + (v_{0,x} - b\omega)t, \quad y_r = b + (v_{0,y} + a\omega)t\,. \quad (42)$$

***Numerical Example* 1.** With $b = -10$ m, $\omega = 0.15$ rad/sec, $v_{0,x} = 0$ m/s and $v_{0,y} = 5$ m/s, the trajectory of the tiny tool thrown by one astronaut is shown in *Figure* 1 of reference [1] (see margin).

***Numerical Example* 2.** With $b = -10$ m, $\omega = 0.95$ rad/sec, $v_{0,x} = -10$ m/s amd $v_{0,y} = 5$ m/s, its trajectory is shown in *Figure* 2 of reference [1] (see margin). Three more trajectories with respect to the rotating frame are also shown in the *Figures* 3 to 5 of reference [1] (see margin).

Suggested Reading

[1] **P Chaitanya *et al*, The Real Effects of Pseudo Forces, *Resonance*, Vol.9, No.6, pp.74–85, June 2004.**

[2] **Angelo Miele, *Flight Mechanics*, Vol.1, Addison-Wesley Publishing Company Inc. Reading, Massachusetts, pp.8–15, 1962.**