In this section of Resonance, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. “Classroom” is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

Solution of Cubic Equations: An Alternative Method

The exact solution of polynomial equations has long been an area of interest. Students are familiar with the solution of linear and quadratic equations, but when they move to cubic and quartic equations, they find that the solution involves a non-intuitive transformation. Cardan’s method of solving the cubic equation is well known (see [1] and [2]; see also [3] for a geometrical interpretation of the method). This article suggests a different and simpler formulation to solve cubic equations.

1. Cardan’s Method

For completeness we give a brief description of Cardan’s method. The general form of the cubic equation is given by

\[ x^3 + ax^2 + bx + c = 0. \] (1)

The first step is to convert this to the ‘depressed cubic’ form by making the substitution \( x = t - a/3 \). This results in the equation:

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Cardan’s method is ingenious and effective, but quite non-intuitive.

\[ t^3 + pt + q = 0, \quad (2) \]

where \( p = b - a^2/3 \) and \( q = (2a^3 - 9ab + 27c)/27 \). The solution of (2) is carried out by making the following transformation:

\[ t = u + v, \quad 3uv = -p, \quad u^3 + v^3 = -q. \quad (3) \]

From these we get a quadratic in \( u^3 \):

\[ u^6 + qu^3 - (p/3)^3 = 0. \quad (4) \]

Solving for \( u^3 \), we get

\[ u^3 = -q/2 \pm \sqrt{(-q/2)^2 + (p/3)^3}. \quad (5) \]

We thus get \( u, v, t \) (and, from them, \( x = t - a/3 \)). Depending on \( p, q \), we get either three real roots or one real root and a pair of complex conjugate roots.

Although Cardan’s method is effective, the logic behind the transformation (3) is not obvious. Other authors have tried the method of ‘completing the cube’ (see [4]), but this method too needs some transformation. The method described below follows a different approach, and should be simpler to understand for beginners.

2. Proposed Method

1. As in Cardan’s method, the first step is to convert the cubic equation to the depressed form, viz.,

\[ x^3 + px + q = 0. \quad (6) \]

2. We now express \( x^3 + px + q \) as a sum in the following way:

\[ x^3 + px + q = \frac{(x + u)^3 + v(x + w)^3}{1 + v}, \quad (7) \]
where \( u, v, w \) are functions of \( p, q \). The motivation behind this is that the equation \((x + u)^3 = -v(x + w)^3\) may be solved simply by taking cube roots of both sides and then solving the resulting linear equation(s) for \( x \). Expanding the numerator of the right hand side of (7), we get:

\[
(1+v)x^3 + 3(u+vw)x^2 + 3(u^2 + vw^2)x + (u^3 + vw^3). \tag{8}
\]

By equating the coefficients of equal powers of \( x \) in (7), we get:

\[
u + vw = 0, \quad u^2 + vw^2 = (1 + v)p/3, \quad u^3 + vw^3 = (1 + v)q. \tag{9}\]

From these we get, successively: \( v = -u/w \), then \( u^2 - uw = (p/3)(1 - u/w) \), and \( u(u - w) = -(u - w)p/(3w) \). Assuming that \( u \neq w \) in general, we get \( w = -p/(3u) \). Next, \( u^3 - uw^2 = q(1 - u/w) \), or, \( u(u^2 - w^2) = -q(u - w)/w \); so, under the same assumption,

\[
u(u + w) = -q/w. \tag{10}\]

Substituting \( w = -p/(3u) \) and simplifying, we get \( u - p/(3u) = 3q/p \), hence:

\[
 u^2 - (3q/p)u - p/3 = 0. \tag{11}\]

This quadratic equation yields:

\[
u = (3q/2p) \pm \sqrt{(3q/2p)^2 + (p/3)}. \tag{12}\]

Here we can use either the positive or the negative square root; and from \( u \) we get \( v \) and \( w \).

3. We now write (7) as

\[(x + u)^3 = -v(x + w)^3. \tag{13}\]

We can find \( x \) by taking the cube root on both sides:

\[
x_1 = -(w + uv^{-1/3})/(1 + v^{-1/3}), \tag{14}\]
where $x_1$ is one of the three roots. The remaining roots $x_2, x_3$ can be found from the following relations between the roots:

$$x_1 + x_2 + x_3 = 0, \quad x_1x_2 + x_2x_3 + x_3x_1 = p. \quad (15)$$

From these we get a quadratic equation which may be solved to give $x_2$ and $x_3$:

$$x_2, x_3 = (-x_1/2) \pm \sqrt{(3/4)x_1^2 - p}. \quad (16)$$

The above relations completely define the solution of the cubic equation.

3. Examples

1. \[x^3 + x + 1 = 0\]

Here $p = q = 1$,

$$u = 1.5 \pm \sqrt{1.5^2 + 1/3} = 3.1073 \text{ or } -0.107.$$  

Let us choose the positive root:

$$v = (3/p)u^2 = 28.966, \quad w = -(p/3u) = -0.1073.$$  

$x_1 = -0.6823$ (real root), $x_2, x_3 = 0.3411 \pm i1.1615$ (complex roots).

2. \[x^3 - 7x + 6 = 0\]

Here $p = -7, \quad q = 6$,

$$u = -1.2857 \pm \sqrt{(1.2857^2 - 7/3} = -1.2857 \pm i0.8248.$$  

Let us choose the negative square root:

$$u = -1.2857 - i0.8248,$$

$$v = (3/p)u^2 = -0.4168 - i0.9090,$$

$$w = -p/(3u) = -1.2857 + i0.8248,$$

giving

$$x_1 = 2.0,$$

$$x_2, x_3 = \frac{-2/2}{(3/4)2^2 + 7.0} = -1.0 \pm 2.0 = 1.0, -3.0.$$  

Suggested Reading


