Classroom

In this section of *Resonance,* we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. “Classroom” is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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Optimization of the Anderson Bridge Experiment

AC bridges are circuits which are often used to measure unknown impedances – resistances, capacitances and inductances. A large number of AC bridges are available for the accurate measurement of impedances. For example, De Sauty Bridge and Schering Bridge are both used to measure unknown capacitances, Owen’s Bridge and Maxwell–Wien Bridge are used to measure inductance. The Anderson Bridge (***Figure*** 1) is used to measure an unknown inductance in terms of a known capacitance and resistance. Anderson’s method is capable of precise measurements of inductance over a wide range of values from a few micro-Henrys to several Henrys. Though the experiment has been a part of the undergraduate curriculum for a long time, most textbooks tend to simply state the balance conditions without discussing the design of the experiment for greater sensitivity. In fact the oldest publication on sensitivity of Anderson’s Bridge is Rayleigh’s paper [1].

Keywords
Anderson Bridge, balance, sensitivity, inductance measurement.

1. Introduction

AC bridges are often used to measure accurately the value of an unknown impedance, for example, self/mutual inductance of inductors or capacitance of capacitors. A large number of AC bridges are available for the accurate measurement of impedances. An Anderson Bridge is used to measure the self inductance of a coil (*Figure* 1). This is an old experiment and has been a part of the undergraduate curriculum for a long time. In fact the oldest publication on sensitivity of AC bridges is Rayleigh's paper [1]. As usually happens in such old subjects, modern textbooks have diluted the attention paid to the details and intricacies of the experiment. Most of the textbooks tend to merely state the balance condition without discussing the design of the experiment for greater sensitivity.

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Over the years, the authors have observed unsatisfactory results reported by the students in terms of accuracy. One reason was an inability to get a mute on the balance condition in the headphone. That is, the human perception of point of minimum rendered results inaccurate. The fact that the human ear perceives in decibels makes the situation worse. A question that invariably arose was whether urbanisation and sound pollution

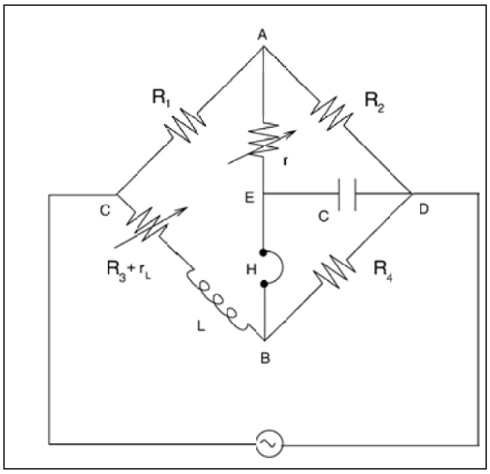

Figure 1. The circuit schematics for Anderson Bridge. The resistance r_L is the resistance of the coil whose self-inductance is being determined.

were the reasons for the inaccuracies of this experiment (physiological constraints) or rather it was ignorance of the relevant physics.

As a test experiment, a careful experimentalist from among the undergraduate students was given basic instructions and a copy of Yarwood's book [2] for review before conducting the Anderson Bridge experiment. *Tables* 1, 2 and 3 list the results reported by this student. The results seem inaccurate and are scattered and deviated from the known value. Also, notice that the student was not able to resolve the minimum for a range of r values (of the order of 50Ω). The results reported by the remaining students of the class by and large suffered from more inaccuracies. Under such circumstances, it warrants a more serious analysis of the experiment. Our

Table 1. The experimentally determined value of inductance (L) in bridge with coil L = 207.4mH and DC balanced resistances R_1 = 560Ω, R_2 = 552Ω and R_4 = 1490Ω.

S.No.	C (μF)	r (Ω)	r_{av} (Ω)	L (mH)
1.	0.05	1000–1050	1025	195.55
2.	0.10	400–450	425	211.00
3.	0.11	390–410	350	217.25
4.	0.12	200–250	225	181.17
5.	0.15	170–200	185	208.45
6.	0.20	43–55	49	196.30

Table 2. The experimentally determined value of inductance (L) in bridge with coil L = 240.2mH and DC balanced resistances R_1 = 560Ω, R_2 = 552Ω and R_4 = 1490Ω.

S.No.	C (μF)	r (Ω)	r_{av} (Ω)	L (mH)
1.	0.05	1250–1310	1280	233.83
2.	0.10	485–515	500	233.52
3.	0.11	390–400	395	222.20
4.	0.15	300–310	305	209.99
5.	0.20	105–115	110	232.92

Table 3. The experimentally determined value of inductance (L) in bridge with coil L = 262.9mH and DC balanced resistances R_1 = 560Ω, R_2 = 552Ω and R_4 = 1490Ω.

S.No.	C (μF)	r (Ω)	r_{av} (Ω)	L (mH)
1.	0.05	1460–1510	1485	264.59
2.	0.10	650–700	675	286.05
3.	0.11	480–510	495	255.22
4.	0.12	390–410	400	244.21
5.	0.20	100–160	130	244.92

analysis is markedly different from that listed by Yarwood [2] and therewith cited Rayleigh's work [1]. However, it utilizes the basic theorems taught in Network Analysis. Our analysis may be viewed as a different perspective on an old problem and an attempt in highlighting the utility of the various Network Theorems in a simple circuit.

Before dwelling on the mathematics and circuit design considerations, for completeness and rendering the article self sufficient, we explain in brief, the Wheatstone Bridge and the standard steps followed for DC and AC balancing of the bridges for determination of the unknown impedances.

Most of the AC bridges are based on a generalised Wheatstone Bridge circuit. As shown in *Figure* 2, the four arms of the DC Wheatstone Bridge are replaced by impedances (Z_{A}, Z_{B}, Z_{C} and Z_{D}), the battery by an AC source and the DC galvanometer by an AC null detector (usually a pair of headphones). Using Kirchoff's Laws, it can be easily shown that the balance or null condition (i.e., when no current flows through the detector, or the potential at the point P becomes equal to that at point R) is given by

$$\frac{Z_{\mathrm{A}}}{Z_{\mathrm{B}}} = \frac{Z_{\mathrm{C}}}{Z_{\mathrm{D}}}. \tag{1}$$

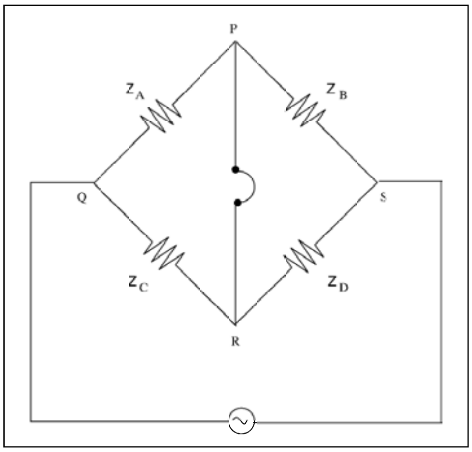

Figure 2. A simple Wheatstone Bridge arrangement.

Equation (1) is a complex equation, i.e., it represents two real equations obtained by separately equating the real and imaginary parts of the two sides. It follows from the fact that both amplitude and phase must be balanced. This implies that to reach the balance condition two different adjustments must be made. That is, the DC and AC balance conditions have to be obtained one by one. The DC balance is obtained using a DC source and moving coil galvanometer by adjusting one of the resistances. After this the battery and the galvanometer are replaced with an AC source and a headphone respectively, without changing the resistances set earlier; the other variable 'impedance' is varied to obtain minimum sound in the headphone.

2. Designing the Circuit

Various combinations of the impedances can satisfy the balance conditions of a bridge. For example, in a DC Wheatstone Bridge using either $R_\text{A} = R_\text{B} = R_\text{C} = R_\text{D} = 10\,\Omega$ or $R_\text{A} = R_\text{B} = 10\,\Omega$ and $R_\text{C} = R_\text{D} = 1000\,\Omega$ would balance the bridge as indicated by (1). However, the bridge is not equally sensitive in both cases. Also interchanging the positions of the source and detecting instrument does not alter the balance conditions (Reciprocity Theorem) but again the bridge may not be equally sensitive in the two cases [2]. The bridge will be more sensitive if, in a balanced bridge, changing the impedance in any one of the arms of the bridge from the value Z required by the balance conditions to value $Z + \delta Z$ (making the bridge off balance) results in a greater current through the detecting instrument. As explained by Yarwood [2], a DC bridge is more sensitive if, *whichever has the greater resistance, galvanometer or battery, is put across the junction of the two lower resistances to the junction of the two higher resistances. Also the unknown resistance should be connected in the Wheatstone bridge between a small ratio arm resistance and a large known variable resistance.* However, it is

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common practice to make all the four arms of a Wheatstone Bridge equal or make them of the same order for the optimum sensitivity. For AC bridges also, the same procedure is followed. Below we discuss how to design an Anderson Bridge of decent sensitivity.

We plan to use various theorems taught in a course on *network analysis* to derive the DC and AC balance conditions of the Anderson Bridge.

Most textbooks derive the DC and AC balance conditions of the Anderson Bridge using Kirchoff's current and voltage laws. As stated earlier, we plan to use various theorems taught in a course on *network analysis* to do the same. Hence, to analyse the bridge, we convert the π network consisting of r, R_2, and X_C and replace it with an equivalent 'T' network, whose three impedances would be given as [3]

$$Z_1 = \frac{rR_2}{r + R_2 + X_C}\ , \qquad (2)$$

$$Z_2 = \frac{R_2 X_C}{r + R_2 + X_C}\ , \qquad (3)$$

$$Z_3 = \frac{rX_C}{r + R_2 + X_C}\ . \qquad (4)$$

The circuit shown in *Figure* 1 can now be viewed as shown in *Figure* 3. The circuit looks more like the fundamental Wheatstone bridge, which purely due to its topology looks trivial and overcomes a mental barrier that a student might have due to a complex looking circuit.

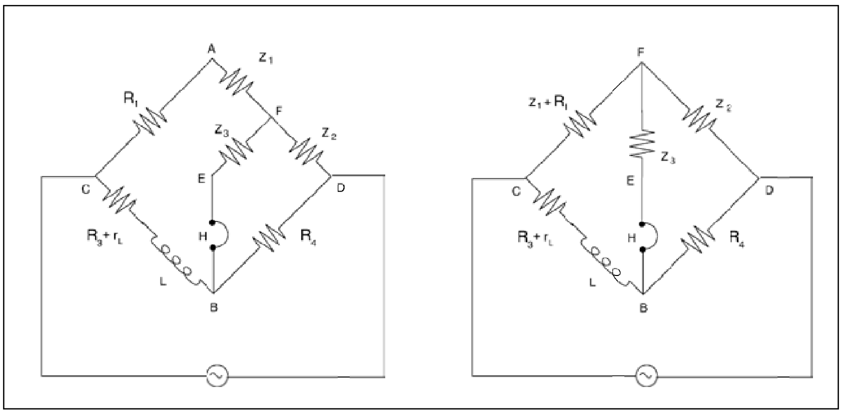

Figure 3. The 'π' shaped circuit between nodes ADE of Figure 1 is replaced by an equivalent 'T' circuit. On rearranging the circuital elements, the complex circuit of Figure 1 reduces to a simple circuit shown here. Along with visual simplicity, it renders the computation of potentials trivial.

2.1 *Balancing Conditions*

For the bridge to be balanced, the current in arm EB should be zero. This demands the potential V_{FB} to be equal to zero. This potential can be worked out as

$$V_{\mathrm{FB}} = V_{\mathrm{F}} - V_{\mathrm{B}} \ ,$$

where, using potential divider expression, we can compute V_{F} and V_{B}. They are expressed as

$$\begin{aligned} V_{\mathrm{F}} &= \left(\frac{Z_2}{R_1 + Z_1 + Z_2}\right) V_{\mathrm{ac}} \ , \\ V_{\mathrm{B}} &= \left(\frac{R_4}{R_3 + r_{\mathrm{L}} + X_{\mathrm{L}} + R_4}\right) V_{\mathrm{ac}} \ , \end{aligned}$$

giving the potential V_{FB} as

$$V_{\mathrm{FB}} = \left(\frac{Z_2}{R_1 + Z_1 + Z_2} - \frac{R_4}{R_3 + r_{\mathrm{L}} + X_{\mathrm{L}} + R_4}\right) V_{\mathrm{ac}} \ . \quad (5)$$

On attaining balance of the bridge

$$\frac{Z_2}{R_1 + Z_1 + Z_2} = \frac{R_4}{R_3 + r_{\mathrm{L}} + X_{\mathrm{L}} + R_4} \ . \quad (6)$$

This is the same as the condition given by (1) with the four impedances replaced by the corresponding impedances in the four arms of *Figure* 3.

Substituting the expressions of (2), (3) and (4) in (6), we have

$$\frac{R_2 X_{\mathrm{C}}}{R_1(r + R_2 + X_{\mathrm{C}}) + rR_2 + R_2 X_{\mathrm{C}}} = \frac{R_4}{R_3 + r_{\mathrm{L}} + X_{\mathrm{L}} + R_4} \ . \quad (7)$$

Collecting and simplifying the real terms of this equation, we get

$$\frac{R_2}{R_4} = \frac{R_1}{R_3 + r_{\mathrm{L}}} \ . \quad (8)$$

This is the DC balance condition of the Anderson Bridge. Now collecting the imaginary terms of (7), we have

$$\frac{L}{C} = \frac{R_4}{R_2}[R_1R_2 + r(R_1 + R_2)] \ . \qquad (9)$$

This is the AC balance condition of the bridge.

For a good audio signal at the headphone, we require that the circuit transfer a substantial amount of power to the headphone.

The unknowns in (8) and (9) are r_L and L respectively. To obtain the DC balance of the bridge, the circuit is made with r shorted, capacitance C open, headphone replaced by a galvanometer and the AC source by a battery or DC power supply. Now R_3 is varied until the galvanometer shows zero deflection. Equation (8) then gives the value of r_{L}. Now using the complete circuit given in *Figure* 1 and leaving the resistances R_1, R_2 and R_3 undisturbed, AC balance is obtained by varying the resistance r. Equation (9) can now be used to compute the value of self inductance L applied in the circuit. Thus the unknown impedance $\sqrt{(L\omega)^2 + r_{\mathrm{L}}^2}$ is computed. For simplicity of circuit design we suggest taking $R_2 = R_4$, which results in $R_1 = r_{\mathrm{L}} + R_3$ from (8).

2.2 *Maximum Power Transfer*

For a good audio signal at the headphone, we require that the circuit transfer a substantial amount of power to the headphone. Since the ear senses the intensity of sound, the headphone as a measuring device is being used as a power meter. Good sensitivity would be attained as large power is being transfered to the headphone for both balanced and unbalanced (AC) condition. Unbalanced AC condition would be represented by replacing r in (9) with '$r + \mathrm{d}r$'.

The Maximum Power Transfer Theorem demands that the load impedance applied to a circuit should be the complex conjugate of the circuit's Thevenin impedance.

The *Maximum Power Transfer Theorem* [3] demands that the load impedance applied to a circuit should be the complex conjugate of the circuit's Thevenin impedance. Since the headphone's impedance is inductive in nature, i.e., it can be represented as '$a + jb$', the theorem demands the Thevenin impedance should be capac-

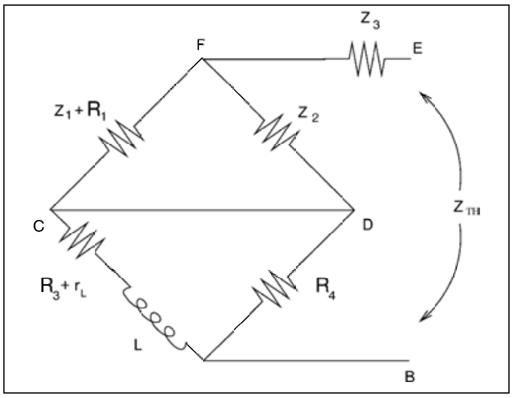

Figure 4. The Thevenin impedance is calculated by shorting nodes 'C' and 'D' and evaluating the impedance as seen from nodes 'EB'.

itive in nature, i.e., of the form of '$a-jb$'. The Thevenin impedance of the Anderson Bridge circuit ($Z_{\rm TH}$) as seen by the headphone is theoretically determined by shorting the AC source (i.e., neglecting the impedance of the source), giving (see *Figure* 4)

$$\begin{aligned} Z_{\rm TH} &= [\{(R_1+Z_1)||Z_2\}+\{(R_3+r_{\rm L}+X_{\rm L})||R_4\}+Z_3] \\ &= \left[\frac{(R_1+Z_1)Z_2}{R_1+Z_1+Z_2}+\frac{R_4(R_3+r_{\rm L}+X_{\rm L})}{R_3+r_{\rm L}+X_{\rm L}+R_4}+Z_3\right] . \end{aligned}$$

Using the DC balance condition given by (8) and imposing $R_2=R_4$, we have

$$Z_{\rm TH} = \left[\frac{(R_1+Z_1)Z_2}{R_1+Z_1+Z_2}+\frac{R_2(R_1+X_{\rm L})}{R_1+X_{\rm L}+R_2}+Z_3\right] .$$

The best way to simplify the above expression is to select $X_{\rm C} \sim 0$. This means the AC audio frequency f should be kept large and the capacitor used should have a very large value of 'C'. From (9), it naturally follows that $X_{\rm L}$ has to be considerably large. Selecting a high audio frequency automatically achieves this. Further, $X_{\rm C} \sim 0$ might be difficult to attain given the fact that frequency is to be in the audible range and very high capacitances are usually electrolytic, thus introducing additional problems. Hence it would be sufficient to select 'C' and 'f' such that $|X_{\rm C}| << |X_{\rm L}|$ and $(rR_1+R_1R_2+rR_2) >> R_2|X_{\rm C}|$. Further, let $R_1 >> R_2$. The justification and advantage of this assumption will be evident later. With these conditions, the Thevenin equivalent impedance reduces to

$$Z_{\text{TH}} = \left[R_2 + \frac{(R_2+r)X_{\text{C}}}{(r+R_2+X_{\text{C}})}\right] . \tag{10}$$

Thus, the basic constraints imposed on the circuit design has made the Bridge's Thevenin equivalent impedance capacitive in nature (equation (10) is of the form a–jb).

$$\begin{aligned} Z_{\text{TH}} &= \left[R_2 + \frac{(r+R_2)|X_{\text{C}}|^2}{(r+R_2)^2+|X_{\text{C}}|^2}\right] \\ &+ \left[\frac{(r+R_2)^2}{(r+R_2)^2+|X_{\text{C}}|^2}\right] X_{\text{C}} . \end{aligned} \tag{11}$$

As stated earlier, for the maximum power to be transferred, the bridge's Thevenin equivalent impedance should be capacitive in nature since the headphone's impedance is inductive in nature. Also, headphones are usually low impedance devices and hence X_{C} and R_2 should be kept as small as possible. While selection of small X_{C} was already imposed, an additional requirement is demanded, i.e., R_2 should also be of small value.

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2.3 *Sensitivity*

To be able to measure the approach of the AC balance point, the sensitivity of the circuit should be good. Here we define the bridge's sensitivity as $(\mathrm{d}P/\mathrm{d}r)$, i.e., the power developed across the headphone should show large change for a small change in r. Upon the application of the *Maximum Power Transfer Theorem*, the expression of current can be written as

$$i = \frac{V_{\text{TH}}}{2\text{Re}(Z_{\text{TH}})} ,$$

where V_{TH} is the Thevenin equivalent voltage. If the Thevenin impedance doesn't match the impedance of the headphone exactly, the current expression for a circuit designed along the lines discussed above reduces to (the Thevenin voltage would be of the same form as that given by (5). Notice $V_{\rm dc}$ has been replaced by $V_{\rm ac}$, the applied AC voltage)

$$i = \left[\frac{1}{\left(R_{\rm H} + \frac{r|X_{\rm C}|^2}{r^2+|X_{\rm C}|^2}\right) + j\left(Z_{\rm H} - \frac{r^2|X_{\rm C}|}{r^2+|X_{\rm C}|^2}\right)}\right] V'_{\rm ac}\ , \quad (12)$$

where $R_{\rm H}$ and $Z_{\rm H}$ are the real and imaginary part of the impedance associated with the headphone being used. The term $V'_{\rm ac}$ is given as

$$V'_{\rm ac} = \left(\frac{Z_2}{R_1 + Z_1 + Z_2} - \frac{R_4}{R_3 + r_{\rm L} + X_{\rm L} + R_4}\right) V_{\rm ac}. \quad (13)$$

Using the constraints imposed and (11), this expression reduces to

$$V'_{\rm ac} = R_2\left[\frac{X_{\rm C}}{R_1(r + R_2 + X_{\rm C})} - \frac{1}{R_1 + X_{\rm L}}\right] V_{\rm ac}\ .$$

Hence, the power dissipated across the headphone is given as

$$\begin{aligned} P &= ii^* R_{\rm H} \\ P &\approx \left[\frac{R_{\rm H}}{\left(R_{\rm H} + \frac{r|X_{\rm C}|^2}{r^2+|X_{\rm C}|^2}\right)^2 + \left(Z_{\rm H} - \frac{r^2|X_{\rm C}|}{r^2+|X_{\rm C}|^2}\right)^2}\right] V'^2_{\rm ac}\ . \end{aligned}$$

Differentiating wrt r (for simplifying the expression we can assume $R_{\rm H}$ and $Z_{\rm H} \sim 0$ which is generally true since headphones are low impedance devices. Further approximation can be made that $V'_{\rm ac}$ is very small and hence is a shallow function of r due to the small values of R_2 and $X_{\rm C}$ in the numerator and large R_1 and $X_{\rm L}$ in the denominator.),

$$\frac{1}{R_{\rm H} V'^2_{\rm ac}}\left|\left(\frac{{\rm d}P}{{\rm d}r}\right)\right| \approx \frac{2}{r^3}\ . \quad (14)$$

The above expression demands r to be small for good sensitivity of the bridge. However, till now we have been silent on the nature of r. Equations (9) will help us understand the effects of our considerations on the nature of r. For a bridge designed along our suggestions, the AC balance condition (9) gives

$$r \approx \frac{L}{R_1 C} \approx X_{\mathrm{C}} \left(\frac{X_{\mathrm{L}}}{R_1} \right) . \qquad (15)$$

In our design, AC balance should be achieved with small r which would result in good sensitivity.

The selection of high frequency, large capacitance with large R_1 makes sure that r, the resistance used to AC balance the bridge, is proportional to the impedance offered by the capacitor. As the case is, in our design AC balance should be achieved with small r which would result in good sensitivity. The interpretations do not vary even if the approximations listed above (14) are not made.

Another design consideration necessary is to make R_1 large. We summarize here the circuit conditions one must maintain to get good sensitivity for the Anderson Bridge:

1. $(R_1 = R_3) >> (R_2 = R_4)$ with R_2 being very small.
2. Select $X_{\mathrm{L}} >> X_{\mathrm{C}}$.
3. Audio frequency should be large.

We put our analysis to test by designing an Anderson Bridge subject to the conditions listed above and determined the self inductance of a given coil. We report some select observations in *Table* 4 to highlight our findings. The headphone used as a current detecting instrument had a resistance R_{H} of about $100\,\Omega$ and inductance

Table 4. This lists the experimentally determined value of inductance (L) in bridge with coil L = 130mH and circuital elements selected as per the analysis described in the text.

S.No.	f (kHz)	$R_1\,(\Omega)$	$R_2\,(\ \ )$	$R_3\,(\Omega)$	C (μ F)	r (Ω)	L (mH)
1.	3	4700	100	4770	0.10	186–188	136.7
2.	3	4700	100	4770	0.01	5570–5720	277.5
3.	1	4700	100	4770	0.10	293–313	192.6

We hope that the present treatment allows the circuit not just to serve as a method of determining self inductance but also becomes an important pedagogical tool for circuit analysis.

$L_{\rm H}$ of about 22 mH. The first reading in *Table* 4 shows the best result, where we ensured that all the conditions required for proper design alongwith Maximum Power Transfer Theorem are satisfied. The ability in determining the balance point, in terms of lowest audio intensity, was also remarkable. For a change of ±1- in the first reading we could detect increase in sound intensity. For the second reading we took a capacitor ten times smaller. This of course results in a ten-fold increase in the values of $X_{\rm C}$ and r. The values of the other impedances selected here also adhered to the listed conditions. However, the inaccuracy in the value of inductance determined is evident. Even the ability to determine the balance point was compromised in the second design. Similar inaccuracy is also evident when the audio frequency is decreased to 1 kHz in the third design reducing the value of $X_{\rm L}$ and increasing that of $X_{\rm C}$.

The design constraints discussed here were not discussed with the experimentalist who reported the results of *Tables* 1, 2 and 3. On examination of the the data, we find that only the first reading of *Table* 2 and *Table* 3 satisfy stated conditions of circuit design. Not only are these the best results in terms of returned value of inductance, but also in terms of ability to resolve the minimum sound.

Conclusion

Based on our derivations using various theorems of *Network Analysis*, we have analysed how to design an Anderson Bridge with good sensitivity. Various considerations show that for a better sensitivity, one should work at high frequencies and use a low R_2 and high capacitance (making $X_{\rm L} >> X_{\rm C}$). We hope that the present treatment allows the circuit not just to serve as a method of determining self inductance but also becomes an important pedagogical tool for circuit analysis.

Suggested Reading

[1] Lord Rayleigh, *Proc. Roy. Soc.*, Vol.49, p.203, 1891.

[2] J H Fewkes and John Yarwood, *Electricity and Magnetism and Atomic Physics*, Vol. I , (2nd ed), Oxford University Press, 1965.

[3] Joseph Edminister and Mahmood Nahvi, *Electric Circuits*, Tata McGraw-Hill, New Delhi, 2002.