Mahlburg's Work on Crank Functions

Ramanujan's Partitions Revisited

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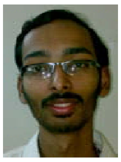

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Karl Mahlburg brilliantly showed the importance of crank functions in partition congruences that were originally guessed by Freeman Dyson. Ramanujan's partition functions are the centre of these works. Not only for the theory on cranks, but for many other researchers in India Ramanujan's work inspired their career in mathematics.

1. Introduction, Background and Motivation

A partition of a positive number n is any non-increasing sequence of positive integers whose sum is n. The partition function $p(n)$ is defined as the number of distinct partitions of a given positive integer n. The rank of a partition is defined to be its largest part minus the number of its parts. Given a natural number n, the number $p(n)$ of ways of partitioning n as a sum of natural numbers seems simple enough to study but turns out to be deceptively difficult. The first few values are

$$p(1) = 1,\ p(2) = 2,\ p(3) = 3,\ p(4) = 5,\ p(5) = 7;$$

these do not seem to give a clue as to either a formula or even how these numbers grow astronomically. For instance, $p(200)$ is almost 4×10^{12}. So, excepting small numbers like $p(5) = 7$ with the partitions

$$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1\,,$$

it would be impossible to enumerate big numbers like $p(200)$ actually. A partition of 4 is $2+1+1$ and corresponding rank is -1 and a partition of 5 is $3+2$ and corresponding rank is 1. If n is a positive integer, then

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two integers a and b are called congruent modulo n, if $a - b$ is divisible by n (they give same remainder when divided by n). It is denoted by $a \equiv b(\text{mod}\, n)$. For example 3462 and 8649 are congruent modulo 1729, i.e., $8649 \equiv 3462(\text{mod}\, 1729)$. Both the numbers 8649 and 3462 give the same remainder when divided by 1729 ($8649 = 1729 \times 5 + 4$ and $3462 = 1729 \times 2 + 4$).

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Combining the idea of partitions and congruences, Ramanujan [1] for the first time introduced the idea of partition congruences that appear in this article. While studying the number of distinct partitions of natural numbers 1 to 200, he observed the following pattern

$$\begin{aligned} p(4),\ p(9),\ p(14),\ldots &\equiv 0\,(\text{mod}\,5) \\ p(5),\ p(12),\ p(19),\ldots &\equiv 0\,(\text{mod}\,7) \\ p(6),\ p(17),\ p(28),\ldots &\equiv 0\,(\text{mod}\,11) \\ p(24),\ p(49),\ p(74),\ldots &\equiv 0\,(\text{mod}\,25) \\ \ldots\ \ \ldots\ &\ \ \ldots \end{aligned}$$

Using the idea of rank of the partition, Freeman Dyson [2] proposed conjectures for the partitions $p(5n+4)$ and $p(7n+5)$. Then for explaining the third conjecture using $p(11n+6)$, he opined that a crank function of the partition would be helpful. Although the actual idea of crank was due to Dyson, while he was pursuing undergraduate course at the University of Cambridge and wanted to explain Ramanujan's congruence modulo 11, a nice definition of crank by Andrews and Garvan [3] is as follows: If $C = C_1 + C_2 + \ldots + C_s + 1 + \ldots + 1$ has exactly r $1s$, then let $o(C)$ be the number of parts of C that are strictly larger than r. The crank is given by

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$$\text{Crank}(C) \;=\; \begin{cases} C_1 & \text{if}\, r = 0\ , \\ o(C) - r & \text{if}\, r \geq 1\ . \end{cases}$$

For example, a partition of 7 is $4+2+1$. Here, $o(7) = 2$ and $r = 1$. Crank of 7 for this partition is 1. Similarly, a partition of 9 is $4 + 3 + 2$ and Crank of 9 for this partition is 4. On the other side, there were some attempts to improve the above partition results for other primes after Ramanujan. Some how research in this direction did not advance much for several decades (until the works of Andrews and Garvan [3], Ahlgren and Ono [4]) and researchers could obtain congruences only for primes less than 33. Mahlburg's outstanding paper improved results that were given by Ono [8] to prove the existence of infinite families of partition congruences for the primes greater than 3.

Ramanujan made substantial contributions in the analytical theory of numbers, elliptic functions, continued functions and infinite series.

Karl Mahlburg's PhD dissertation work [5] solidly established the role of cranks (originally conceptualized by Dyson in 1944 and details are in Section 3) in the theory of partition congruence functions. His profound theory shows that indeed crank plays a central role in explaining partition congruences of Ramanujan-type and it has once again opened discussions on congruence modulo functions by Ramanujan. Mahlburg's theory not only gave general ideas on crank functions but showed a novel way to generate them through crank generating functions. He proved Ken Ono's conjecture [6], which has connections to the earlier works by Andrews and Garvan [3], Dyson [2] and Ramanujan [1]. Ramanujan made substantial contributions in the analytical theory of numbers, elliptic functions, continued functions and infinite series. His conjectures in mathematics are still being researched and new theories are being added. For a list of ongoing research associated with Ramanujan's work and theories named after him, see an article by Ken Ono in the *Notices* [7].

Proceedings of the National Academy of Sciences (PNAS) awarded the Paper of the Year Prize, 2005 to Karl Mahlburg, then a doctoral candidate in mathematics at the University of Wisconsin, Madison. Mahlburg's award winning paper in PNAS was entitled 'Partition congruences and the Andrews–Garvan–Dyson crank.' The paper gave a theoretical backing to the cranks and proved very interesting results through crank generating functions. The focus of this article is to understand Mahlburg's work on crank and give readers a short account of the developments on crank to study Ramanujan-type functions and also highlight how researchers in India are inspired by Ramanujan's work in general. In Section 2 we present Euler's theorems on partitions and Rogers–Ramanujan identities. Section 3 begins with Dyson's guesses on crank theory and a very brief outline of this theory that was taken to new heights by Mahlburg.

2. Euler's Theorems and Rogers–Ramanujan Identities

A generating function is a formal power series whose coefficients encode information about a sequence (a_n), where $n \in N$. The generating function of a sequence (a_n) is given by

$$F(a_n; x) = \sum_{n\geq 0} a_n x^n.$$

In a similar way, the generating function of a sequence $(a_{n_1,n_2,\ldots,n_k})$, where $n_1, n_2, \ldots, n_k \in N$ is

$$F(a_{n_1,n_2,\ldots,n_k}; x_1, x_2, \ldots, x_3) = \sum_{n_1\geq 0} \sum_{n_2\geq 0} \cdots \sum_{n_k\geq 0} a_{n_1,n_2,\ldots,n_k} x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}.$$

The partition function has a nice generating function:

$$\sum_{n=0}^{\infty} = \prod_{r=1}^{\infty} \frac{1}{1-q^r},$$

where the convention is to put $p(0) = 1$. The above identity is formally 'seen' to be true as an identity in q by expanding each term of the right-hand side as a geometric series. Indeed, the identity can be proved analytically when $|q| < 1$ as the infinite product converges. For $|q| < 1$, note that

$$\Pi_{n\geq 1}(1-q^n)^{-1} = \Pi_{n\geq 1}(1 + q^n + q^{2n} + \ldots).$$

A typical term of the expanded product is of the form q^n with $n = r_1n_1 + r_2n_2 + ... + r_kn_k$ for some $n_1 < n_2 < ... < n_k$ and $r_1, ..., r_k \in N$. Thus, the coefficient of q^n is the number $p(n)$ of partitions of n. Since a power series determines its coefficients, one may compare coefficients of like powers of q to prove the above formula. Indubitably, the following result discovered by L Euler would rank high among the most elegant of mathematical results in most people's lists.

Euler's 'Odd versus Distinct' Theorem

The number $p_{\mathrm{odd}}(n)$ of partitions of n into odd numbers is also the number of partitions $p_d(n)$ of n into distinct numbers.

Proof. Similar to

$$\Pi_{n\geq 1}(1-q^n)^{-1} \;=\; \sum_{n\geq 0} p(n)q^n$$

one obtains

$$\Pi_{n\geq 1}(1-q^{2n-1})^{-1} \;=\; \sum_{n\geq 0} p_{\mathrm{odd}}(n)q^n.$$

Consider the infinite product $\Pi_{n\geq 1}(1+q^n) = p_d(n)q^n$. Again, it is evident from expanding this product that a power q^n occurs as many times as n can be written as a sum $n_1 + n_2 + ... + n_k$ of distinct natural numbers for some k. In other words,

$$\Pi_{n\geq 1}(1+q^n) \;=\; p_d(n)q^n.$$

Now,

$$\Pi_{n\geq 1}(1+q^n) \;=\; \frac{\Pi_{n\geq 1}(1-q^{2n})}{\Pi_{n\geq 1}(1-q^n)} = \frac{1}{\Pi_{n\geq 1}(1-q^{2n-1})}.$$

This completes the proof.

We mention one more identity which can be proved by actually giving a bijection.

The number of partitions of n into exactly m parts equals the number of partitions of n into parts in which the largest part is m. Therefore, the number of partitions of n into at most m parts equals the number of partitions of n into parts in which each part is at most m.

Proof. The proof is by plotting an array of points corresponding to a partition in the following manner. For a partition $n_1+n_2+...+n_r = n$, where $n_1 \geq n_2 \geq ... \geq n_r$, draw an array consisting of dots with n_1 dots in the first row, n_2 dots in the second row (centered to the left), etc. Associate to this array, the 'conjugate array' obtained by counting column-wise. For instance, $8 = 2 + 2 + 4$ gives the conjugate array corresponding to the partition $8 = 3 + 3 + 1 + 1$.

The operation of conjugation produces the bijection we are looking for.

The same graphic method can be used to prove:

If $p_d^e(n)$, $p_d^o(n)$ denote, respectively, the partitions of n into an even number of distinct parts and an odd number of distinct parts, then $p_d^e(n)-p_d^o(n) = (-1)^r$ if $n = r(3r \pm 1)/2$ and is 0 otherwise.

The numbers $n(3n - 1)/2$ are known as the pentagonal numbers. The above partition-theoretic identity has the following interesting reformulation:

Euler's Pentagonal Theorem

$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} = \Pi_{r=1}^{\infty}(1 - q^r)$ *for* $|q| < 1$.

Similarly, the following wonderful identities have reformulation in terms of the partition functions.

Rogers–Ramanujan Identities

If $|q| < 1$, *then*

$$1 + \sum_{n\geq 1} \frac{q^{n^2}}{(1-q)...(1-q^n)} \quad = \quad \Pi_{n\geq 0} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

and

$$1 + \sum_{n\geq 1} \frac{q^{n(n+1)}}{(1-q)...(1-q^n)} \quad = \quad \Pi_{n\geq 0} \frac{1}{(1-q^{5n+3})(1-q^{5n+4})}.$$

These identities are equivalent forms of:

(i) *The number of partitions of* n *into parts, any two of which differ by at least* 2, *equals the number of partitions of* n *into parts congruent to* ± 1 *modulo* 5.

(ii) *The number of partitions of* n *into parts* > 1, *any two of which differ by at least* 2, *equals the number of partitions of* n *into parts congruent to* ± 1 *modulo* 5.

G Jacobi proved some beautiful theorems two of which we recall here.

Jacobi's Triple Product Identity

If $t \neq 0$ *and* $|q| < 1$, *then*

$$\Pi_{n\geq 0}(1-q^{2n+2})(1+tq^{2n+1})(1+t^{-1}q^{2n+1}) \quad = \quad \sum_{n\in Z} t^n q^{n^2}.$$

Remarkably, Jacobi's triple product identity yields Euler's pentagonal theorem on replacing q by $q^{3/2}$ and t by $-q^{1/2}$.

Another beautiful result due to Jacobi is:

$$\Pi_{n\geq 1}(1-q^n)^3 \quad = \sum_{r\geq 0}(-1)^r(2r+1)q^{r(r+1)/2}.$$

Combining Euler's pentagonal theorem and the above theorem of Jacobi, it is quite easy to prove:

$$p(5n+4) \quad \equiv \quad 0 (\mathrm{mod}\, 5)\ .$$

3. Evolving of Mahlburg's Theory on Cranks

Mahlburg's approach to crank generating functions together with Klein forms is novel.

Mahlburg's outstanding work in 2005 on crank functions led to new aspirations for further developments in the theory. He showed that his theory on crank functions satisfies a wider range of congruences than those predicted by earlier researchers like Ahlgren, Andrews, Garvan, Dyson, and Ono. His approach to crank generating functions together with Klein forms is novel. In this section, we begin with a brief description of Ramanujan's congruences which actually led Dyson to initiate work on cranks.

Ramanujan's Congruence Functions and Origins of Dyson's Crank

Ramanujan's congruences for the partition function $p(n)$ in Section 1 can be shortened to,

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5, \\ p(7n+5) &\equiv 0 \pmod 7, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

These congruences can be written succinctly in the form $p(ln - \theta_l) \equiv 0 \pmod l$, if we define $\theta_l = (l^2 - 1)/24$ (see Ahlgren and Ono [4]). Ramanujan first proved two congruences in his original paper [1] and later in 1921 (after his death in 1920), G H Hardy extracted proofs of all the three congruences from an unpublished manuscript of Ramanujan. Later, [8] published proofs for the congruences after correcting them and also proved Ramanujan's congruence for the modulus 11^s for some arbitrary s. Lehner [9,10] established further properties of congruence modulo 11. In 1944, Dyson published a note 'Some guesses in the Theory of Partitions', where he has given some combinatorial arguments to the above partition functions. In this work he introduced notation $N(m,k)$ to denote the number of partitions of k with rank m and $N(m,q,k)$ to denote the number of partitions of k whose rank is congruent to m modulo q. He obtained the rank of the partition by subtracting the number of parts in a partition from the largest part. According to his example, ranks of the partitions of k are $k-1, k-3, k-4, ..., 2, 1, 0, -1, -2, ..., 4-k, 3-k, 1-k$ and

$$N(m,q,k) \;=\; \sum_{x=-\infty}^{\infty} N(m+xq,k).$$

He conjectured that the partitions of $5k+4$ and $7k+5$ are divided respectively into five and seven equal classes. Notationally these conjectures are stated as

$$\begin{aligned} N(0,5,5k+4) &= N(1,5,5k+4) \\ &\quad ,...,N(4,5,5k+4)\ . \\ N(0,7,7k+5) &= N(1,7,7k+5) \\ &\quad ,...,N(6,7,7k+5)\ . \end{aligned}$$

By using the principle of conjugacy, we can see that $N(m,k) = N(-m,k)$ and $N(m,q,k) = N(q-m,q,k)$. Dyson used this principle and reduced the statements of the above conjectures into smaller independent identities. To simplify computing procedures Dyson translated these identities into algebraic forms by the means of generating functions. Instead of stating a similar conjecture for the partitions of $11k+6$, Dyson predicted that conjecture with modulus 11 is false and introduced a hypothetical 'crank' of partition, $M(m,q,k)$, which explains the number of partitions of k whose crank is congruent to m modulo q.

By using the relation $M(m,q,k) = M(q-m,q,k)$, Dyson guessed that

$$\begin{aligned} M(0,11,11k+6) &= M(1,11,11k+6) \\ &= M(2,11,11k+6) \\ &= M(3,11,11k+6) \\ &= M(4,11,11k+6)\ . \end{aligned}$$

After Dyson's Crank

Atkin and Swinnerton-Dyer [11] proved conjectures of Dyson, and Atkin [12] extended Ramanujan's famous congruences to arbitrary powers of 5, 7, and 11. Working on Dyson's cranks, Andrews and Garvan gave a formal definition to the crank (see Section 2). Ken Ono has been researching on Ramanujan's contributions and it is evident from his article in *Notices* [7] that he is a fan of Ramanujan's work since his senior high school days. Ono [6] and Ahlgren and Ono [4] advanced the idea on cranks and gave further generalizations. Ono [6] in his seminal work, established that it is possible to express every prime ≥ 5 in the Ramanujan-type congruences, followed by Ahlgren and Ono [4], who laid complete theoretical framework to describe such congruences.

Theorem (Ono)

Let $l \geq 5$ be prime and let k be a positive integer. A positive proportion of the primes m have a property that

$$p\left(\frac{l^k m^3 n + 1}{24}\right) \equiv 0 \pmod{l}$$

for every non negative integer n coprime to m.

Theorem (Ahlgren and Ono)

Define the integer $x_l \in \{\pm 1\}$ by $x_l := \left(\frac{-6}{l}\right)$ for each prime $l \geq 5$ and let S_l denote the set of $(l+1)/2$ integers $S_l := \left\{y \in (0, 1, ..., l-1) : \left(\frac{y+\theta_l}{l}\right) = 0 \text{ or } -x_l\right\}$, where $\theta_l = (l^2-1)/24$. Then Ahlgren and Ono proved the monumental result for general partition functions: If $l \geq 5$ is prime, k is a positive integer, and $y \in S_l$, then a positive proportion of the primes $I \equiv -1 \pmod{24l}$ have the property that

$$p\left(\frac{I^3 n + 1}{24}\right) \equiv 0 \pmod{l^k}$$

for all $n = 1 - 24y \pmod{24l}$ with $(I, n) = 1$.

Mahlburg [5] in his award-winning work improved Ono's results for the primes ≥ 5. By using crank generating functions (defined by Andrews and Garvan [3]) and elegantly linking them to Klein forms he has demonstrated that the role of crank plays indeed a central role in understanding the Ramanujan-type congruences. Statement of his theorem is as follows: *Let $l \geq 5$ be primes and let i and j be positive integers, then there are infinitely many arithmetic progressions $Ak + B$ such that $M(m, l^j, Ak + B) \equiv 0 \pmod{l^i}$ simultaneously for every* $0 \leq m \leq l^j - 1$.

We believe that Mahlburg's extraordinary paper has inspired many undergraduate and graduate students. Some of them might look into Ramanujan's work published almost a century ago and develop very good research problems. We hope there will be further developments in crank functions by adopting other kinds of generating functions and proving alternative results to that of Mahlburg. We look for more papers that discover links between Ramanujan's mathematics and present day science.

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