The Scientific Enterprise

10. The Role of Mathematics

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Two Kinds of Descriptions

When we say that the sea is blue and the sky is high, our descriptions are *qualitative*. We become aware of the qualitative features of the world primarily through our sense perceptions. Some of these features have pleasing effects on us. We recognize colors and smells, distinguish the hard from the soft, the smooth from the rough, etc. We have all enjoyed serene sunsets, fragrant flowers, and magnificent music.

The *quantitative* aspects of the world refer to numbers and to mathematical relationships that can be associated with things and processes. For example, heights of mountains, distances of stars, the amount of rainfall in a region in the course of a year, wind speed: all these involve quantitative descriptions.

Generally speaking, but not always, it is possible to associate a quantitative side with a qualitative description. Physics is primarily interested in accounting for phenomena in terms of the quantitative aspects of the world.

The lad in the story (Antoine de St. Exupéry, *The Little Prince*-1943, Mariner Books, 2000) complains that grown-ups are so much obsessed with figures that when they are told that one has made a friend, rather than ask about his hobbies and interests, they are eager to find out about his age, the number of people in his family, the amount of money his father makes, etc. It is fair to say that this is a very good description of the physicist's attitude to the world of phenomena. For example, this is how the poet Valmiki describes the Moon "in the sky shining with rays, obtaining the middle portion of sky. Spreading a great quantity of moon shine, the moon looked like a bull in heat among cows… the moon comes up destroying the sins of the world, causing the great ocean to grow and causing all living beings to shine. …" (*Sundara Kanda*, Canto V). However, the same moon will be described quite differently by the scientist who may say, for example, that it is some 384,404 km from the Earth, that its sidereal day is 27.322 days, that it reflects only about a fourteenth part of the light it receives from the Sun, etc. Each description has its own contextual relevance.

The aim of physics is to give coherent explanations for phenomena and processes, but these must first be described. Descriptions involve adjectives: good, bad, big, small, hot, cold, ugly, beautiful, strong, weak, fast, slow, etc. Often such descriptions can be expressed on a comparative scale. Thus we say that something is larger, prettier, hotter, etc., than something else. Once the comparison is established we may ask, "By how much?" This question is not always answerable. For example, we can say by how much the area of New Delhi is greater than that of Chennai, but we cannot say by how much the music of Thyagaraja is more (or less) beautiful than that of Tansen, or by how much the philosophy of Shankara is more (or less) profound than that of Charvaka.

Whenever the question, *By how much?* can be answered unequivocally we have a situation where measurement is possible. In such cases we associate numbers with the physical situation considered. Then we are dealing with a metric aspect of the world, a physical quantity.

Many such quantities arise in the course of everyday life. The volume of a bottle, the weight of a baggage, the speed of a train, the time it takes to read this page: all these are examples of physical quantities. While investigating the world, the physicist finds it convenient to introduce many physical quantities. Some of these have entered the common vocabulary because of the role they play in technological societies. Thus we speak of 92.5 megahertz on the radio dial, the 75 watt lamp, and a temperature of 30 degrees celsius.

On the Nature of Mathematics

Mathematical thought is as ancient as civilization. There is hardly a concept developed by the human mind that has stood the test of centuries or achieved the degree of universal acceptance as mathematical truths. Yet, even prolific creators in the field have had sharp disagreements as to the ultimate nature of their subject. This is because the origins of the most fundamental concepts of mathematics are not easy to trace. As the mathematician Leopold Kronecker perceptively observed, said: "God made the integers, all the rest is the work of man" (*Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk,* Jahresberichte der Deutschen Mathematiker Vereinigung, Vol.2, 1893.)

But mathematicians have also tried to make integers. That is to say, they have tried to define the concept of the most fundamental elements of arithmetic. For example, from the primitive concepts of 'set', 'natural numbers', 'successor', and 'belong to', as well as five axioms for the natural numbers, one has tried to derive the existence and properties of all kinds of numbers. (Bertrand Russell, *Introduction to Mathematical Philosophy* (1919), reprinted: Routledge, London, 1993.) But such attempts have not always been very successful. After all, it is difficult to define the number one without using a noun in the singular; but this already implies a knowledge of *one*.

A second difficulty is that mathematics is intimately linked to the logical functioning of the brain, and we know as yet very little about how this process comes about. Mathematical reasoning is such a perfect logical process that in a sense erring in mathematics should be quite a feat for the brain. And yet, our brains do this all the time, because the human brain is equally capable of deviating from strict logic. The capacity to err, which computers don't yet have, enables imagination, art, literature, poetry, myth-making, and the like.

Then again, though mathematical reasoning is logical, mathematical creation is not. Without creativity, we could not have advanced much beyond counting with our fingers. We don't have any idea of how creativity emerges. Analytical geometry is a systematic approach to certain problems. But the idea of representing pairs of numbers as points on a plane (the basic idea in analytical geometry) is said to have come to René Descartes all of a sudden in 1619. Or, consider the theory of groups which has turned out to be one of most valuable tools of modern physics. It required the perceptive genius of Évariste Galois to recognize it in the roots of algebraic equations. Sudden awareness of the least apparent properties of numbers, the instant discovery of profound mathematical truths that would take pages and pages to establish, and uncanny capacity for split second calculations, have all been recorded. They defy easy explanation. We search in vain to recognize the source of the genius of a Leonard Euler or a Srinivasa Ramanujan.

Intuition also propels mathematical thinking as effectively as logic guides it. If a branch of mathematics grows fruitful as a tree, its seed is begotten by an act of creation, fertilized by intuition and made to grow by logic. Many schools of thought have emerged, such as the axiomatic, the intuitionist, and the logical schools, each presenting itself as a cornerstone in the foundations of mathematics (Howard Eves, *Foundations and Fundamental Concepts of Mathematics,* Dover Publications, New York, NY: 1990).

Mathematics and Deductive Logic

Logic is an indispensable tool in every kind of reasonable idea, let alone intellectual discourse. Logic is used by mathematicians to *derive and demonstrate* their results, by propagandists to *persuade* others to their point of view, by theologians to *prove* the existence of God, and sometimes by philosophers to *disprove* the contentions of their opponents.

The power of logical reasoning is enhanced through the use of symbols, just as the scope of language is greatly increased by writing. If logic enriches mathematics by being embedded in it, mathematics has enriched logic immensely through its symbolism.

Mathematics is closely allied to deductive logic, where one begins with a set of premises (postulates), and deduces their various consequences by the use of abstract reasoning. These conclusions are implicit in the postulates themselves, and the process of reasoning merely extracts out, as it were, those conclusions. Therefore, as long as the basic postulates are valid (or taken as such), so will the conclusions be.

For example, if we take the Euclidean postulate of parallels to be true, then all the theorems of Euclidean geometry are valid. If, however, we take another set of postulates (such as that more than one parallel is possible to a straight line through an external point), then we will be led to another set of theorems whose validity will be perfectly consistent with the accepted postulates. The question as to which set of postulates is correct has no relevance in mathematics, because postulates are, by definition, what we take to be correct.

Inductive and Deductive Logic and Science

Recall that in physics (more generally in science) we adopt a very similar procedure in the so-called hypothetic-theoretic method. Here we begin with a set of hypotheses (corresponding to the postulates of mathematics), and deduce from them a number of consequences. However, here one spells out conditions for the validity or otherwise of the hypotheses. Specifically, if the conclusions from the hypotheses are consistent with the observed world, we are inclined to believe in the correctness of the hypotheses from which they follow. If not, we discard the hypotheses in question.

When hypotheses are formulated in mathematical symbolism, many deductions can be precisely and effectively derived. This is the key to the success of any physical theory. Indeed, there is no theory in physics that does not have a quantitative (mathematical) basis. From gravitation to electromagnetism, from relativity to the Bohr model, from the de Broglie hypothesis to the quark model, every physical theory has a mathematical aspect which alone permits one to explore all its manifold consequences. The most insightful or interesting (hypothetical) declaration about the physical world (such as, for example, that there are thought waves) has no relevance in physics unless it can be formulated in a quantitative manner.

At this point we may note an important difference between the formulation of postulates in mathematics and that of hypotheses in physics. In the first instance, the goal is to simply explore the theorems that follow from the postulates. These theorems may have nothing to do with the physical world, and may serve no practical need whatever, although such may turn out to be the case eventually. For example, when Bernard Riemann explored the consequences of a non-Euclidean postulate, he did not have the faintest notion that some day this could become relevant in the context of General Relativity. The same may be said of group theory, matrix algebra, and many other branches of mathematics.

In physics, however, the conclusions drawn are clearly recognized: the details of a well-defined phenomenon or the data of observations and experiments. The goal of the theory builder is to come up with a set of hypotheses from which, by pure logical deductions, one can arrive at conclusions that correspond to what have been empirically established. In a sense, it is a kind of 'back-calculation,' but where a fundamentally new conceptual framework is introduced. Thus, a hypothesis in physics may be regarded as a provisional postulates.

Aside from the hypothetic-deductive method, there is also another mode of arriving at conclusions in physics. This is by drawing broad generalizations on the basis of a great number of carefully observed events. Suppose that certain randomly chosen elements *a*, *b*, *c*, *d* and *e* from a set consisting of a great many elements possess a certain property. One concludes from this that all elements of the set (including those that have not been observed) have this property. This mode of reasoning constitutes inductive logic. From the fact that copper, silver, iron, mercury, all expand when heated, if one concludes that all metals expand when heated, one does inductive reasoning. In other words, in deductive logic we go from the general to the particular, whereas in inductive logic we go from particular statements to general ones.

Important Difference in Truth Content

Now it is important to note a fundamental difference in the conclusions between deductive and inductive reasoning. All the conclusions in deductive logic are completely valid (assuming that the postulates are so), while the conclusions in inductive logic are never one hundred percent reliable even if the data on which they are based are faultless. This is because deductive logic is pure reason whereas inductive logic is careful conjecture.

It is important to note this because here lies a fundamental difference between the modern scientific method of interpreting the world and ancient scientific modes. Consider, for example, the ancient (often religious) views on the nature or origin of the physical world. In any tradition, these matters are based on the utterances or writings of certain individuals who are deemed infallible. Everything that follows from or is consistent with those utterances or writings must therefore be accepted as true. This is an example of deductive reasoning. However, unlike in mathematics, here the practitioners don't see (or are willing to accept) that the postulates are arbitrary, or have little relation to empirical facts or figures. Unlike in the hypothetic-theoretic system, one is not prepared to discard the basic premises even if the consequences lead to patent contradictions with observed facts.

We may now ask why there is such a fundamental difference between deductively acquired knowledge (which is completely true) and inductively acquired knowledge (which is true only with a certain degree of probability). At one time, this fact used to be given as an argument against the use of induction. The point to note is that deductive knowledge is not new knowledge. It does not tell us anything about the world that is not already contained in the basic premises. We say that deductive knowledge is *analytic*. Inductive statements, however, are quite different. They tell us something about the world of which we are as yet unaware. Inductive knowledge is *synthetic*. (These terms were introduced by the eighteenth century philosopher Immanuel Kant.) A degree of uncertainty is the price we pay for getting something new. (In this context, see Hans Reichenbach, *The Rise of Scientific Philosophy*, University of California Press, Los Angeles, CA, 1951.)

Scope and Limits of Logic

Since logic is the indispensable structure on which both physics and mathematics rest, it should be of some interest to inquire into the scope and limits of logic itself. These have been explored by many thinkers in great detail. Suffice it to say that in spite of its enormous power and potential, logic is not always as trustworthy as those committed to the scientific methodology may be inclined to think.

Logic is not very helpful when we try to establish its own reliability. For, if you think it is reliable, you do not need any proof of it; and if you do not think it is reliable, you should not accept a proof! The unreliability of the use of logic in a system can be brought out if it can be shown that mutually inconsistent conclusions can be drawn within that system.

Consider, for example, the famous statement of Epimenides of Crete to the effect that every inhabitant of his town was a liar. It is impossible to decide if this is true or false since the speaker himself is an inhabitant of the town. Similarly, the pair of sentences: “The next statement is true; the previous statement is false”, constitute a proposition whose parts make sense, but which as a whole is neither true nor false. (See, for example, N Rescher, *Paradoxes: Their Roots, Range, and Resolution.* Open Court Publishing, 2001).

All this leads to the suspicion that logic or what often appears as logical reasoning, may not always be a reliable ingredient of a language, a fact that should be clear to anybody who listens to political speeches and certain other types of discourses with any care.

Certain theorems in mathematics, due to Kurt Gödel, showed that the inner consistency of logical systems may not always be provable. (Ernest Nagel, and James R Newman, *Gödel’s Proof,* New York University, New York, 1958.) This is tantamount to assuring us that the whole intellectual structure of science is based on foundations whose reliability cannot be fully guaranteed! This insight seems to be present in the writings of some ancient philosophers who insisted that logic is at best an interesting game that the mind can play, but is not to be counted upon as a useful instrument in the search for the Absolute Truth.

This somewhat negative appraisal of logic (popular among anti-science intellectuals and some proponents of spiritual truths) may be valid at the theoretical philosophical level, but it fails to see the indispensability of logic in meaningful and consistent interpretations of the world in a good many contexts of everyday concern, and the role that such interpretations have played in the attainment of universally acceptable relative truths. Truths proclaimed on the basis of direct experience and with doubtful logical support, however convincing they may be to the *experiencer*, are of little value to others beyond provoking sighs of admiration or reverence or sympathy. In order for truths to be shared, for insights to be further explored, and for statements to be rendered reconcilable we need logic, both deductive and inductive, with all their intrinsic constraints. Just as the underlying chaos of the subconscious supports all human behavior, both sane and otherwise, the shaky ground on which logic seems to rest does not deter its significant role in physics and mathematics, two of the most sophisticated and fruitful fields of human intellectual endeavor.

Mathematics and the Dynamic Aspects of the Physical World

The physical world is throbbing with activity. Nothing ever remains the same. Change is the one pervasive feature in the universe. This inescapable fact has been recognized from the most ancient times. The Heraclitan phrase that all is flux and nothing is stationary has its echo in the thoughts and reflections of all cultures. In certain Hindu worldviews, all that is ephemeral is unreal, only that is Real which does not change. This is as much a definition as a deep insight. However, from this to reject or trivialize ephemeral reality can have and has had serious undesirable effects on the advances that individuals and civilization can make.

If the physical world were a single static permanent entity, cold and consummate, unchanging and cast for all eternity, there would be no physics or chemistry, no planet or life, no past or future. Such perhaps was the state prior to the breaking of the perfect symmetry that gave rise to the Big Bang of creation (in current cosmology) from which, we suspect, emerged this physical universe. It is this dynamic aspect of the world that interests the physicist. With the incessant passage of time, each view, every facet and part of the physical universe from the microcosmic to the galactic, transforms into yet another.

Practically every change results from, and also serves as the cause of other changes. In other words, there is not only change, but also mutual and relative change. The length of a rod changes with change in temperature; the velocity of a body is affected by the forces acting on it; an electromotive force is produced in a circuit when the magnetic flux through it changes; increasing gravitational pressures lead to extremely high temperatures, which in turn provoke nuclear fusion reactions, and so on.

Since physics explores the quantitative aspects of the world, it is also concerned with the mathematical aspects of changes. Observable changes are expressed in terms of mathematical variables. Indeed, the phenomenal world as replete with invisible physical quantities that are continuously changing with respect to one another. To understand the nature of physical phenomena therefore we need to study the relative changes of measurable quantities.

The branch of mathematics that deals with mutual changes is differential calculus. Hence its importance in physics. If we lived in a static world, there would be little need for calculus to explain it. The dynamic aspects of the world cannot be fully grasped without using the calculus. Then again, quantitative investigation of the physical world would be impossible without conceptual physical entities: that is, quantities that may not be directly amenable to sense perceptions. Consider, for example, the motion of a particle. All that we observe is change of position with the passage of time. From this fact of observation we define *average speed* as the ratio of the distance traveled to the time taken. The body does not experience such a thing as speed. When we are moving in a uniformly moving vehicle we are not aware of speed, fast or slow. However, in the scientific analysis of motion the concept of speed becomes valuable. This is only one example of the countless mathematical concepts that serve the physicist.

Aside from fundamental quantities such as length, time, mass and electric charge, every other physical quantity is a construct of the human mind. These measurable constructs are ingenious codes to describe complex situations and the multifold processes occurring in the physical world. Consider a familiar definition and explore some of its consequences. Instantaneous velocity is defined as, $\mathbf{v} = \mathrm{d}\mathbf{r}/\mathrm{d}t$, where $\mathbf{r}$ is the position vector of a particle, and the acceleration $\mathbf{a}$ is defined as $\mathrm{d}^2\mathbf{r}/\mathrm{d}t^2$. We may now express Newton's second law of motion as: $\mathbf{F}= m\mathbf{a}$ or $\mathbf{F} = m\mathrm{d}^2\mathbf{r}/\mathrm{d}t^2$.

This is an example of a differential equation. We see here how the formulation of a physical law in mathematical terms leads to a differential equation. This will always be the case when we express a physical process or law that involves mutual changes in physical quantities. Thus arises the importance of differential equations in physics.

In the example considered above, only two variables are involved: time and the particle's position. However, it may happen that in a physical situation more than two variables come into play. In such cases we are led to partial differential equations in the mathematical formulation of the laws. Consider, for example, the study of the properties of gases. Here, at least four variables have to be taken into account: mass, volume, pressure, and temperature. Consequently, when the gas laws are expressed in mathematical forms, we get partial differential equations, such as Maxwell's relations in thermodynamics.

As an extreme example of multivariable systems, consider the changes occurring in a continuous medium, such as in a vibrating string. Every element of the string has a position which is changing with time. In fact, we have an infinite number of points on the string. Here again, we may expect a partial differential equation. We may also look upon the problem simply in terms of three variables: the position of a point on the string, its displacement from the mean position, and time.

Many such situations arise in our investigation of the physical world. A system which consists of a continuum rather than of a discrete number of particles is referred to as a field. Generally speaking, the equation describing the evolution of a field, i.e. how a particular attribute of the field changes from point to point and from instant to instant, is given by a partial differential equation. Much of classical theoretical physics may be reduced to the study and solutions of various field equations (partial differential equations). This is equally true of wave mechanics: quantum mechanics treated in terms of Schrödinger equation and its extrapolations.

From the mathematical point of view, there is no essential difference between change with respect to time (d/dt) or change with respect to space (d/dx). It is simply a matter of taking the derivative with respect to one or the other variable. The notion of the dynamic aspect of the physical world may also be extended to include changes of characteristics from point to point in a system or medium; or even with respect to variables that do not involve space or time at all, as with the thermodynamic variables of a gas.

It is clear that the role of mathematics in physics extends beyond the tagging of numbers to the measurable features of the world. Mathematics becomes indispensable in the analysis of the manner in which the various quantitative attributes of physically observable systems vary, and vary with respect to one another. Such analyses will have two major consequences: First, they will enable us to determine the precise functional dependence between or among the variables involved. This, in effect, may also be a reflection in the mathematical equations, and is intimately connected with causality.

Second, they will also permit us to calculate and determine how these quantities change from point to point and from instant to instant. In other words, the mathematical formulation of the precise manner in which mutually influencing changes occur in the context of a given phenomenon will enable us to describe its evolution or development with a fair amount of precision. That is to say, the application of mathematics to the dynamic aspects of the world will *enable us to predict* the evolution of systems. And this is no mean accomplishment. Indeed, in the history of human thought no other mode of prediction has proved to be as unfailing and precise as the ones that emerge in mathematical physics and computational astronomy.

Mathematics and Ordinary Human Languages

In the seventeenth century Galileo famously said that the book of the universe is written in mathematical language (*Il libro dell'Universo è scritto in lingua matematica*). This statement has been repeated many times. But it must be noted that there are similarities and well as important differences between mathematics and ordinary languages. The goal of any language is to communicate. But communication can also occur without language, as illustrated in the *mudras* of Bharata Natyam.

First let us consider some similarities between ordinary languages and mathematics. The essential characteristics of a language are vocabulary which are clearly defined terms, conjunctions which are connecting elements, grammar which consists of rules for proper use of terms, syntax: which specifies the order in which words are to be used. Finally, every language has a community of mutually understanding members. In mathematics too there are well defined terms (laplacian, divergence, integral, set, etc.), conjunctions (operations) grammar (logic), and syntax (associativity, commutativity), and community (mathematically initiated persons)..

As to differences: Unlike ordinary languages, mathematics cannot convey feelings and emotions like anger, love, contempt, etc. Translations from the physical world to mathematics is never one to one. Thus $2 + 5 = 7$ can be translated in many ways: two mangoes plus five mangoes are equal to seven mangoes, etc. Or again, the linear second order differential equation may be interpreted as expressing an LCR circuit or damped oscillations. Also, the articulation (the way they are pronounced) of written mathematical symbols is different in different spoken languages, as with Chinese ideograms. Thus $1 + 2 = 3$ is pronounced: Eins und zwei machen drei (German), un et deux font trios (French), oNNum renDum mooNu (Tamil), ek aur do theen (Hindi), one and two make three (English), and so on. Then again, reasoning (logic) is far more important in mathematics than in ordinary languages.

Perhaps the most important difference between ordinary languages and mathematics is the power of the latter. When a phenomenon is expressed in mathematical terms, it enables us to unravel hidden aspects of the physical world. The planets Neptune and Pluto were discovered this way, as also the existence of electromagnetic waves. It was mathematical exploration that led to the discovery of the existence of the positron, the Ω^- and several other elementary particles.

The mathematical formulation of processes or situations enables one to predict the evolution of a system: something that it is simply impossible to do in any purely verbal language. The role of differential equations is significant in such contexts.

It is seldom recognized that but for mathematical physics humanity would never have known about the laws governing the microcosm. Not even the most powerful microscopes imaginable can reveal atomic transitions or why the water molecule is stable.

We thus see that a good deal of scientific knowledge, especially of the physical, chemical and microcosmic aspects of the physical world, could not have come about without the use of mathematics. The truths that art and literature, religion and mysticism reveal are significant in their own right, but they are of a totally different nature from those derived from instrument-based and mathematically explored science.

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