Madhava, Gregory, Leibnitz, and Sums of Two Squares

Shailesh A Shirali

Shailesh Shirali heads the Community Math Centre in Rishi Valley School and works in the field of teacher education. He is the author of many books and articles in mathematics, written for interested students in the age range 13–19 years. He also has a close involvement with the mathematical olympiad movement in the country.

This article describes a connection between the Gregory–Leibnitz series for pi and the representations of positive integers as sums of two squares. It emerges from a study of the geometry of lattice points in the plane.

1. The Mādhavā–Gregory–Leibnitz Series for π

The identity generally called the *Gregory–Leibnitz series*,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots, \qquad (1)$$

was first proved by the Kerala mathematician Mādhavā in the fourteenth century (see [3] for details). Here we describe a curious connection between the Mādhavā series and the representations of positive integers as sums of two squares; it emerges from a study of the geometry of lattice points in the plane. Like so many other things in mathematics, it was first pointed out by Gauss. (See [2] for another account of this connection.)

The connection between the Madhava series and representation of integers as sums of two squares was first noted by Gauss.

There are many ways of proving (1), the simplest being term by term integration of both sides of the following identity from $x = 0$ to $x = 1$,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^{n-1}x^{2n-2} + (-1)^n \frac{x^{2n}}{1+x^2},$$

followed by showing that

$$\int_0^1 \frac{x^{2n}}{1+x^2}\,\mathrm{d}x \to 0, \quad \text{as} \quad n \to \infty.$$

Keywords
Gregory–Leibnitz series, lattice points, sums of two squares, Gauss circle problem.

Another proof comes from evaluating $\int_0^{\pi/4} \tan^n x \, dx$, using integration by parts.

2. Lattice Points Within a Circle

Let $\mathcal{C}(n)$ denote the circle with radius $\sqrt{n}$, centered at the origin, $(0,0)$, and let $f(n)$ denote the number of lattice points within this circle. An equivalent definition for $f(n)$, more convenient for computational purposes, is: $f(n)$ *is the number of pairs* (x,y) *of integers for which* $x^2 + y^2$ *does not exceed* n.

$f(n)$ is the number of lattice points within the circle $\mathcal{C}(n)$ which is centered at the origin and has radius $\sqrt{n}$.

It seems plausible that $f(n)$ is approximately equal to the area of $\mathcal{C}(n)$, i.e., that $[f(n)/(\pi n)]$ is close to 1, when n is large. The figures bear this out convincingly:

n	πn	$f(n)$	$f(n)/\pi n$
10^1	31.4159	37	1.17775
10^2	314.159	317	1.00904
10^3	3141.59	3149	1.00236
10^4	31415.9	31417	1.00003
10^5	314159.3	314197	1.00012
10^6	3141592.6	3141549	0.99999

The figure $f(10) = 37$ may be checked by hand calculation.

To formally prove that $[f(n)/(\pi n)] \to 1$ as $n \to \infty$, we associate a 1×1 square with each lattice point in $\mathcal{C}(n)$, centered at that lattice point, with sides parallel to the coordinate axes. *Figure* 1 shows this for the case $n = 11$, with the squares shown shaded. The total area of the shaded region is clearly $f(n)$.

Within each of these unit squares, the point furthest from the center is a corner, its distance from the center being $\frac{1}{\sqrt{2}}$. Therefore, the circle with center $(0,0)$ and radius $\sqrt{n} + \frac{1}{\sqrt{2}}$ completely encloses the shaded region,

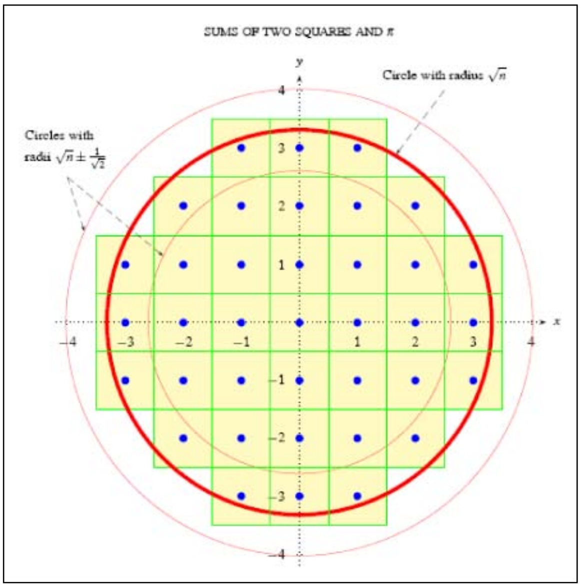

Figure 1. Lattice points contained within $\mathcal{C}(n)$, for n = 11; the three circles have radii, $\sqrt{n} - \frac{1}{\sqrt{2}}$, $\sqrt{n}$, and $\sqrt{n} + \frac{1}{\sqrt{2}}$, respectively; the smallest circle lies completely within the shaded region, and the largest circle completely encloses this region.

and the circle with center $(0, 0)$ and radius $\sqrt{n} - \frac{1}{\sqrt{2}}$ lies completely within the shaded region. Hence:

$$\pi \left(\sqrt{n} - \frac{1}{\sqrt{2}} \right)^2 \leq f(n) \leq \pi \left(\sqrt{n} + \frac{1}{\sqrt{2}} \right)^2 . \quad (2)$$

It follows that

$$1 + \frac{1}{2n} - \sqrt{\frac{2}{n}} \leq \frac{f(n)}{\pi n} \leq 1 + \frac{1}{2n} + \sqrt{\frac{2}{n}}, \quad (3)$$

and therefore that $[f(n)/(\pi n)] \to 1$ as $n \to \infty$.

Relation (2) yields simple bounds for the error term, $f(n) - \pi n$:

$$\left| f(n) - \pi n - \frac{\pi}{2} \right| < \pi \sqrt{2n}. \quad (4)$$

The problem of representing integers as sums of two squares was first studied by the Greek mathematician Diophantus.

3. Which Numbers are Representable as Sums of Two Squares?

The problem of representing integers as sums of two squares is a very well studied topic, the first known results being due to the Greek mathematician Diophantus (third century AD). Let n be a positive integer. It is known that:

Fermat used the principle of infinite descent to show that a prime number of the form 4*k*+1 is expressible as a sum of two squares.

1. If $n \equiv 3 \pmod 4$, then n cannot be written as a sum of two squares. Example: 7 is such a number. More generally, if n contains a prime factor $p \equiv 3 \pmod 4$ raised to an *odd* power, then n cannot be written as a sum of two squares. Example: $78 = 2 \times 3 \times 13$ is such a number.

2. If n is a prime number of the type $1 \pmod 4$, then n can be written as a sum of two squares in precisely one way. Example: $13 = 2^2+3^2$. (We do not consider $13 = 3^2 + 2^2$ as distinct from $13 = 2^2 + 3^2$.) In this case the total number of pairs (x, y) of integers such that $x^2+y^2 = n$ is 8. For example, for $n = 13$ we get the pairs $(\pm 2, \pm 3)$ and $(\pm 3, \pm 2)$. This result is due to Fermat, who proved it using the principle of infinite descent (see [3]). It is one of the gems of elementary number theory, and quite a challenge to prove.

3. For the general case, when n is an arbitrary positive integer, a formula for the number of pairs (x, y) of integers such that $x^2 + y^2 = n$ was first found by Jacobi. It is expressed in terms of the number of divisors of n, thus: *Let the number of divisors of n of the type $1 \pmod 4$ be $d_1(n)$, and let the number of divisors of n of the type $3 \pmod 4$ be $d_3(n)$. Then the number of pairs (x, y) of integers such that $x^2+y^2 = n$ is equal to $4\,[d_1(n) - d_3(n)]$.*

   Example: Take $n = 65$. The divisors of 65 are $1, 5, 13, 65$. These are all of the form $1 \pmod 4$, so $d_1(n) = 4$, $d_3(n) = 0$, $4\,[d_1(n) - d_3(n)] = 16$. The integer pairs (x, y) for which $x^2+y^2 = 65$ are $(\pm 1, \pm 8)$, $(\pm 8, \pm 1)$, $(\pm 4, \pm 7)$, $(\pm 7, \pm 4)$, and these are indeed 16 in number.

   Or take $n = 39$. The divisors of 39 are $1, 3, 13, 39$, so that $d_1(n) = 2$, $d_3(n) = 2$, $d_1(n) - d_3(n) = 0$. And

Jacobi's theorem for f(n) allows us to write it as a summation.

indeed there are no pairs (x, y) of integers for which $x^2 + y^2 = 39$.

Jacobi used elliptic functions and 'q-series' to prove his identity (see [4] for details), so his was certainly not an elementary proof.

4. A Formula for $f(n)$

Jacobi's result allows us to write $f(n)$ as a summation:

$$f(n) \ = \ 1 + 4\sum_{k=1}^{n} [d_1(k) - d_3(k)] \, ,$$

the '1' coming from the representation of 0 as $0^2 + 0^2$. Let us write this as

$$f(n) \ = \ 1 + 4\sum_{k=1}^{n} d_1(k) - 4\sum_{k=1}^{n} d_3(k) \ , \qquad (5)$$

and see what each sum on the right might signify. Consider first the quantity

$$\sum_{k=1}^{n} d_1(k) \ = \ d_1(1) + d_1(2) + d_1(3) + \cdots + d_1(n).$$

For convenience we introduce a symbol $a_{i,j}$ to codify divisibility:

$$a_{i,j} \ = \ \begin{cases} 1, & \text{if } i \text{ divides } j, \\ 0, & \text{if } i \text{ does not divide } j. \end{cases} \qquad (6)$$

Then $d_1(k) = a_{1,k}+a_{5,k}+a_{9,k}+a_{13,k}+\cdots = \sum_{i\equiv 1 \pmod 4} a_{i,k}$, and so:

$$\sum_{k=1}^{n} d_1(k) \ = \ \sum_{k=1}^{n} \ \sum_{i\equiv 1 \ (\mathrm{mod}\ 4)} a_{i,k}.$$

As only finite summations are involved, we may freely interchange the order of summation:

$$\sum_{k=1}^{n} d_1(k) \ = \ \sum_{i\equiv 1 \ (\mathrm{mod}\ 4)} \ \sum_{k=1}^{n} a_{i,k}.$$

The quantity $\sum_{k=1}^{n} a_{i,k}$ is easily computed: it is simply the number of multiples of i not exceeding n; hence

$$\sum_{k=1}^{n} a_{i,k} \;=\; \left\lfloor \frac{n}{i} \right\rfloor, \tag{7}$$

where $\lfloor z \rfloor$ denotes the largest integer not exceeding z. It follows that

$$\sum_{k=1}^{n} d_1(k) \;=\; \left\lfloor \frac{n}{1} \right\rfloor + \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{9} \right\rfloor + \left\lfloor \frac{n}{13} \right\rfloor + \cdots .$$

In the same way we get:

$$\sum_{k=1}^{n} d_3(k) \;=\; \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{7} \right\rfloor + \left\lfloor \frac{n}{11} \right\rfloor + \left\lfloor \frac{n}{15} \right\rfloor + \cdots .$$

We thus get an unexpected formula for the number of lattice points enclosed by $\mathcal{C}(n)$:

$$\begin{aligned} f(n) \;&=\; 1 + 4\Bigg(\left\lfloor \frac{n}{1} \right\rfloor - \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{5} \right\rfloor \\ &- \left\lfloor \frac{n}{7} \right\rfloor + \left\lfloor \frac{n}{9} \right\rfloor - \left\lfloor \frac{n}{11} \right\rfloor + \cdots \Bigg). \end{aligned} \tag{8}$$

Since $[f(n)/n] \to \pi$ as $n \to \infty$, it follows that

$$\begin{aligned} \lim_{n\to\infty} \; \frac{1}{n}\Bigg(\left\lfloor \frac{n}{1} \right\rfloor - \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{5} \right\rfloor - \left\lfloor \frac{n}{7} \right\rfloor \\ + \left\lfloor \frac{n}{9} \right\rfloor - \left\lfloor \frac{n}{11} \right\rfloor + \cdots \Bigg) = \frac{\pi}{4}. \end{aligned} \tag{9}$$

This identity serves as a nice companion to the Mādhavā–Gregory–Leibnitz result quoted earlier!

5. Closing Remarks

5.1 *Gauss Circle Problem*

Relation (4) implies the following relation:[1]

$$f(n) - \pi n = O(n^{1/2}). \tag{10}$$

[1] The 'big *O*' notation used here has the following sense: $f(x) = O(g(x))$ if there exists a number $M > 0$ such that $|f(x)| < M\,|g(x)|$ for all sufficiently large values of x. For example, $x^3 + 2x^2 = O(x^3)$, $\sin x = O(1)$, ; $x \ln x = O(x^2)$.

The Gauss circle problem is regarded as difficult, and remains open.

The proof of (10) being quite easy, we may feel we can get better results by using stronger methods. But this turns out to be much more challenging than anticipated!

The determination of the least value θ of c such that $f(n) - \pi n = O(n^c)$ is known as the *Gauss circle problem.* From the above we have $\theta \leq 1/2$. A literature search reveals that successively tighter upper bounds for θ have been obtained over the decades: 1/3 (Voronoi, 1903; Sierpiński, 1906), 17/53 (Vinogradov, 1935), and more recently, 131/416 (Huxley, 2003). In the other direction it is known that $\theta \geq 1/4$ (Hardy; Landau). In [5], the determination of θ is described as a "very difficult problem", and it remains open; see [6].

5.2 *The Three Squares Problem*

Much less is known about the three-dimensional version of this problem. The function now under study is the cardinality $g(n)$ of the set $S(n) = \{(x,y,z) \mid x,y,z \in \mathbb{Z},\ x^2+y^2+z^2 \leq n\}$. For example, $g(5) = 57$, and $g(10) = 147$. Here we expect that $g(n)$ is close to $\frac{4}{3}\pi n^{3/2}$ for large n, and the figures bear this out. Define $h(n) = g(n) \div \frac{4}{3}\pi n^{3/2}$; then we have:

$h(100) \approx 0.99528,\ h(1000) \approx 0.99992,\ h(10000) \approx 0.99978$

By associating a cube of side 1 with each point $(x,y,z) \in S(n)$ and arguing as earlier, we may show that

$$\frac{4}{3}\pi\left(\sqrt{n} - \frac{\sqrt{3}}{2}\right)^3 \leq g(n) \leq \frac{4}{3}\pi\left(\sqrt{n} + \frac{\sqrt{3}}{2}\right)^3,$$

and deduce that $g(n) \div \frac{4}{3}\pi n^{3/2} \to 1$ as $n \to \infty$.

Further analysis along these lines becomes difficult, as there is no simple formula for the number of integer triples (x,y,z) for which $x^2+y^2+z^2 = n$. As earlier, we may ask for the least value τ of c such that $g(n) - \frac{4}{3}\pi n^{3/2} = O(n^c)$. Some upper bounds obtained for τ are:

2/3 (Vinogradov, 1963), 29/44 (Chamizo and Iwaniec, 1995), 21/32 (Heath–Brown, 1999). But this problem too remains open.

Suggested Reading

[1] S A Shirali, Euler, Nilkantha and pi, *Resonance*, Vol.2, No.5, pp.29–43, 1997.

[2] D Hilbert, and S Cohn-Vossen, *Geometry and the Imagination*, New York, Chelsea, pp.27–39,1999.

[3] S A Shirali, Fermat's Two-Square Theorem, *Resonance*, Vol.2, No.3, pp.69–83, 1997.

[4] Eric W Weisstein, *Sum of Squares Function*. From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/SumofSquaresFunction.html

[5] G H Hardy and E M Wright, *An Introduction to the Theory of Numbers*, 5th ed. Oxford, England, pp.267–272, 1979.

[6] Eric W Weisstein, *Gauss's Circle Problem*, From MathWorld – A Wolfram Web Resource. http://mathworld.wolfram.com GausssCircleProblem.html

Address for Correspondence
Shailesh A Shirali
Rishi Valley School
Rishi Valley 517 352
Andhra Pradesh, India
Email:
shailesh.shirali@gmail.com