# Swinging in Imaginary Time

## More on the Not-So-Simple Pendulum

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**When the small angle approximation is not made, the exact solution of the simple pendulum is a Jacobian elliptic function with one real and one imaginary period. Far from being a physically meaningless mathematical curiosity, the second period represents the imaginary time the pendulum takes to swing upwards and tunnel through the potential barrier in the semi-classical WKB approximation[1] in quantum mechanics. The tunneling here provides a simple illustration of similar phenomena in Yang–Mills theories describing weak and strong interactions.**

[1]When the exact quantum mechanical calculation of the tunneling probability for an arbitrary potential $U(x)$ is impracticable, the Wentzel–Kramers–Brillouin approach, invented independently by all three authors in the same year as the Schrödinger equation, provides an approximate answer.

**Keywords**
Pendulum, Jacobi elliptic functions, tunneling, WKB, instantons.

## 1. Introduction

Consider a point mass $m$ at the end of a rigid massless stick of length $l$. The acceleration due to gravity is $g$ and the oscillatory motion is confined to a vertical plane. Denoting the angular displacement of the stick from the vertical by $\phi$ and taking the gravitational potential to be zero at the bottom, the constant total energy $E$ is given by

$$E = \frac{1}{2}ml^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 + mgl(1-\cos\phi) = mgl(1-\cos\phi_0), \quad (1)$$

where $\phi_0$ is the maximum angle. Isolating $(\mathrm{d}\phi/\mathrm{d}t)^2$ and taking its square root gives $\phi$ as an implicit function of the time $t$ through

$$t = \sqrt{\frac{l}{2g}}\int_0^{\phi}\frac{\mathrm{d}\phi}{(\cos\phi-\cos\phi_0)^{1/2}}$$

$$= \sqrt{\frac{l}{4g}} \int_0^{\phi} \frac{\mathrm{d}\phi}{\left(\sin^2(\phi_0/2) - \sin^2(\phi/2)\right)^{1/2}} \; . \qquad (2)$$

Following the treatment in Landau–Lifshitz [1], we define $\sin\xi = \sin(\phi/2)/\sin(\phi_0/2)$, in terms of which the period $T$ becomes

$$\begin{aligned} T &= 2\sqrt{\frac{l}{g}} \int_0^{\phi_0} \frac{\mathrm{d}\phi}{\left(\sin^2(\phi_0/2) - \sin^2(\phi/2)\right)^{1/2}} \\ &= 4\sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\xi}{(1 - k^2 \sin^2 \xi)^{1/2}} \; . \qquad (3) \end{aligned}$$

This can be used to define $K(k)$, the complete elliptic integral of the first kind, via

$$T \equiv 4\sqrt{\frac{l}{g}} K(k) \; . \qquad (4)$$

When $k^2 = (\sin(\phi_0/2))^2$ can be neglected, one recovers the period $2\pi\sqrt{l/g}$ of the simple small angle pendulum. Note that the standard elliptic function notation $K(k)$ is a bit misleading: $K$ is really a function NOT of $k$, but of $k^2$. Hence for $k^2 \ll 1$, we can consistently keep $k$ as a small non-zero quantity while neglecting $k^2$. The substitution $y = \sin\xi$ shows that (3) can also be written as

$$\begin{aligned} K(k) &= \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\xi}{(1 - k^2 \sin^2 \xi)^{1/2}} \\ &= \int_0^1 \frac{\mathrm{d}y}{((1 - y^2)(1 - k^2 y^2))^{1/2}} \; . \qquad (5) \end{aligned}$$

## 2. Elliptic Functions and Integrals

Let us convert the definite integral (5) into an indefinite one by replacing the upper limit 1 by $y$. Also introducing the dimensionless time variable $x = t\sqrt{g/l}$, this yields

$$x = \int_0^y \frac{\mathrm{d}y'}{((1-y'^2)(1-k^2y'^2))^{1/2}} \quad . \tag{6}$$

The right-hand side of (6) is known as the Legendre elliptic integral of the first kind and is denoted by $F(\arcsin y, k)$. The inverse of $F$, implicitly defined by (6), is the Jacobian elliptic function $y = \mathrm{sn}x$ [2]. Assembling all this, the exact time dependence of the angle is found to be

$$\phi(t) = 2\arcsin\left(\sin\left(\sqrt{\frac{g}{l}}t\right)\sin\frac{\phi_0}{2}\right) \quad , \tag{7}$$

which becomes the familiar

$$\phi(t) = \phi_0 \sin\left(\sqrt{\frac{g}{l}}t\right) \tag{8}$$

in the limit $\sin\phi_0 \approx \phi_0$.

This is an appropriate point to mention an essential difference between trigonometric and elliptic functions: Considered as functions of the complex variable $z = x + iy$, the former are bounded and have a real period along the $x$-axis, but blow up exponentially as $y$ goes to positive or negative infinity, while the latter have an additional imaginary period along the $y$-axis. We will have to state this and a few other properties of elliptic functions without proof, as the required complex analysis is a bit involved. We refer the interested reader to Mathews and Walker [2], whose treatment we mostly follow. For $\mathrm{sn}x$, we have already met the real period $K(k)$ in (4); the imaginary period $2K'$ is defined via

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**Box 1. Elliptic Functions**

A function $f(z)$ is said to be doubly periodic in the complex $z$-plane with periods $\omega_1$ and $\omega_2$ if $f(z + n\omega_1 + m\omega_2) = f(z)$ for any pair of integers $n, m$. Double periodicity along two independent directions requires the ratio $\omega_1/\omega_2$ to have an imaginary part. Thus the function maps a single parallelogram cell $\Pi$ defined by $\omega_1$ and $\omega_2$ to the entire $z$-plane. If the singularities of $f(z)$ consist of poles, the resulting meromorphic doubly-periodic function is called an elliptic function. By Liouville's theorem, $f(z)$ must have poles in $\Pi$ if it is not to be a constant. Since the parallel sides of the parallelogram $\partial\Pi$ bounding $\Pi$ give equal and opposite contributions, and $f(z)$ is meromorphic, $\oint_{\partial\Pi} f(z)\mathrm{d}z = 0$ by Cauchy's theorem. This means the residues of the poles in $\Pi$ must add up to zero. The simplest way to get vanishing total residue is either to choose two simple poles with opposite residues, or one double pole in $\Pi$. These correspond to the Jacobi and the Weierstrass elliptic functions, respectively. The latter, denoted by $\wp$, is easier to represent explicitly (in the form of an infinite double sum over all lattice points), but our present problem of a planar pendulum involves the Jacobi elliptic functions (interestingly, the Weierstrass ones appear in the treatment of the spherical pendulum!). The lattice is generated by $\omega_1 = K$ and $\omega_2 = iK(k')$ that are defined in (5) and (9). The double periodicity properties follow by integrating (6) in the complex $y$-plane along different contours: $\mathrm{sn}(2K - y) = \mathrm{sn}y$ is proven by choosing a path that goes from 0 to $y$ to 1 along the real axis, encircles 1 in a clockwise sense and comes back to $y$ on a path going left just below the real axis. Using $\mathrm{sn}(-y) = -\mathrm{sn}y$ which follows from (6), we have $\mathrm{sn}(2K+y) = -\mathrm{sn}y$. Shifting the argument again by another $2K$, we get $\mathrm{sn}(4K+y) = +\mathrm{sn}y$. Extending the contour from 1 to $1/k$, clockwise encircling $1/k$ and coming back to $y$ on a path just below the real axis yields $\mathrm{sn}(y + 2iK') = \mathrm{sn}y$ on account of (9) and the fact that for $1 < x < 1/k$, $\sqrt{1 - x^2} = \pm i\sqrt{x^2 - 1}$.

Historically, the study of elliptic functions originated around the end of the seventeenth century in the study of arc lengths of ellipses and a curve called Bernoulli's lemniscate, leading to the integral $\int_0^\alpha \mathrm{dx}/\sqrt{1 - x^4}$. Count Fagnano presented an addition theorem for this integral in 1750. Euler extended the theorem to integrals with $\sqrt{1 + ax^2 + bx^4}$ in the denominator. Legendre's studies of more general integrands such as $E + F/\sqrt{Q(x)}$, where $E$ and $F$ are rational functions and $Q$ is a cubic or quartic polynomial culminated in his 1832 *Treatise on Elliptic functions and Euler integrals*. In 1828 Abel and Jacobi almost simultaneously defined new functions by inverting such integrals and showed that they were doubly-periodic. The terminology was also changed: these functions are now called elliptic, while Legendre's elliptic functions are referred to as elliptic integrals. Jacobi's elliptic functions can also be expressed as the ratios of quadratic combinations of Jacobi $\Theta$-functions that are quasiperiodic rather than periodic in the complex plane. The so-called Ramanujan $\Theta$-function is a further generalization of Jacobi $\Theta$-functions, but it is not due to Ramanujan himself. Ramanujan introduced what he called "mock $\Theta$-functions", which are nowadays regarded as a class of 'modular forms', defined by their transformation properties under modular transformations of the complex plane. Such a transformation takes $z$ to $(az + b)/(cz + d)$, where $a, b, c, d$ are integers and $ab - cd = 1$. For more on Ramanujan's remarkable work in this area we refer the reader to the book *Ramanujan's lost notebook*, George E Andrews and Bruce C Berndt, Part I, Springer, New York, 2005.

$$K' = \int_1^{1/k} \frac{\mathrm{d}y}{((y^2-1)(1-k^2y^2))^{1/2}} \quad . \tag{9}$$

The double-periodicity property means that

$$\mathrm{sn}(z+4K) = \mathrm{sn}(z+2iK') = \mathrm{sn}z \ . \tag{10}$$

It can be shown that $K'(k) = K(k')$ if $k'^2 = 1 - k^2$ by going over to the variable $z = \sqrt{(1-k^2y^2)/(1-k^2)}$ in (9). This 'duality relation' will be useful in relating the pendulum's behavior near the top and bottom points.

## 3. The Physical Meaning of the Imaginary Period

We will now show that the imaginary period $K'$ coincides with the interval of imaginary time the pendulum spends while 'swinging' from $+\phi_0$ to $-\phi_0$ *not* via the bottom point $\phi = 0$ as in its normal motion, but by continuing beyond $+\phi_0$ to the *top* point $\phi = \pi$ and all the way down in the same direction to $\phi = 2\pi - \phi_0$, which is of course the same position in space as $-\phi_0$. During this 'motion', the total energy $mgl(1-\cos\phi_0)$ is below the potential energy $mgl(1-\cos\phi)$, making the kinetic energy $\frac{1}{2}ml^2(\mathrm{d}\phi/\mathrm{d}t)^2$ negative; hence the time $t$ may be regarded as imaginary. Although we started with a familiar classical mechanics problem, this is clearly reminiscent of quantum tunneling, which raises the question of what an *ab initio* WKB-like treatment of the problem without referring to elliptic functions would reveal.

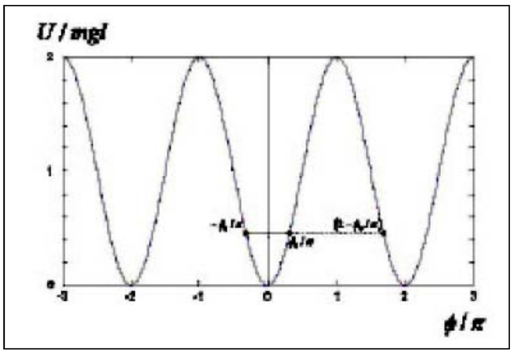


***Figure 1.** The real and imaginary 'paths' are most clearly understood by referring to the potential. The solid line represents the real time oscillations, while the dotted one corresponds to imaginary time tunneling.*

Specifically, we wish to calculate the imaginary time $T'$ that the mass takes to cover the above described path. To do this, all we have to do is to change the limits in the integral (2), giving

$$T' = \int_{\phi_0}^{2\pi-\phi_0} \frac{\mathrm{d}\phi}{(\mathrm{d}\phi/\mathrm{d}t)} = 2\sqrt{\frac{l}{g}} \int_{\phi_0}^{\pi} \frac{\mathrm{d}\phi}{\left(\sin^2(\phi_0/2) - \sin^2(\phi/2)\right)^{1/2}}, \tag{11}$$

where in the last equation we have used the symmetry of the potential around $\phi = \pi$. In the new integration range $\pi \geq \phi \geq \phi_0$, $\sin\xi = \sin(\phi/2)/\sin(\phi_0/2) \geq 1$. Setting $y = \sin\xi$ as before gives

$$T' = 2i\sqrt{\frac{l}{g}} \int_1^{1/k} \frac{\mathrm{d}y}{\sqrt{(y^2-1)(1-k^2y^2)}} = 2i\sqrt{\frac{l}{g}} K'(k) \ . \tag{12}$$

We note that $i = \sqrt{-1}$ and the $y$-direction period $K'(k)$ given in (9) naturally appear! The second, purely imaginary period of the Jacobi elliptic function indeed coincides with the WKB tunneling time.

## 4. Instantons, Small Oscillations, Duality

The duality relation $K'(k) = K(k') = K(\sqrt{1-k^2})$ mentioned earlier allows us to identify and interrelate some interesting limiting cases in the pendulum's behavior. Our problem is non-linear, and it happens to share some important aspects of the also non-linear Yang–Mills field theory. In the $k^2 \approx 0$ small-amplitude limit, both our pendulum and Yang–Mills fields can be treated perturbatively, meaning just a few lowest powers in the field amplitudes ($\phi$ in our case) need be considered. An example in the pendulum problem is the first anharmonic term used in [3]. However, in both cases, the imaginary time behavior of the small or even zero amplitude limit contains rich non-perturbative phenomena related to the real time perturbative regime via duality. For

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**Box 2. Yang–Mills Fields and their Instantons**

According to the 'Standard Model' of fundamental particle physics, strong, electromagnetic and weak interactions are mediated by gauge fields. The familiar electric and magnetic fields interact only with charges and currents but not directly with themselves, resulting in a linear theory. The Yang–Mills gauge fields used for describing strong and weak interactions, however, have self-interactions, and are fundamentally non-linear. The pendulum considered here is a surprisingly useful toy model for illustrating novel features (such as instantons) of Yang–Mills fields that arise from non-linearity. For our purposes, it will be sufficient to consider the simplest Yang–Mills gauge fields as electric and magnetic fields with an additional internal symmetry index $a$ that runs from 1 to 3; thus we now have $\mathbf{E}_a$ and $\mathbf{B}_a$ instead of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$. Just as the electromagnetic fields can be obtained from the potentials $\mathbf{A}$ and $\Phi$ via $\mathbf{E} = -\dot{\mathbf{A}} - \nabla\Phi$ and $\mathbf{B} = \nabla \times \mathbf{A}$ ($c$, the speed of light is taken as unity here), the Yang–Mills fields are obtained from potentials $(\mathbf{A}_a(\mathbf{r},t), \Phi_a(\mathbf{r},t))$, albeit in a more complicated and non-linear way. The potentials are the analogues of the 'coordinate' $\phi(t)$ and the 'electric fields' $\mathbf{E}_a$ the analogues of the 'velocity' $\dot{\phi}$. The field energy density is proportional to $\mathbf{E}_a \cdot \mathbf{E}_a + \mathbf{B}_a \cdot \mathbf{B}_a$, where, pursuing the analogy, the first and second terms can be identified as 'kinetic' and 'potential' energies, respectively (the Einstein convention of summing over the repeated index $a$ is understood). When one switches to imaginary time to investigate tunneling, $\mathbf{E}_a \to i\mathbf{E}_a$, and the energy density becomes $-\mathbf{E}_a \cdot \mathbf{E}_a + \mathbf{B}_a \cdot \mathbf{B}_a$. This allows non-vanishing field configurations obeying $\mathbf{E}_a = \pm\mathbf{B}_a$ with zero energy. Such fields are said to be self-dual or anti-self-dual, and the equation essentially says (square root of the kinetic energy) = $\pm$ (square root of the potential energy), just as in (15). In particular, there are (anti) self-dual fields that vanish over all space at $t = -\infty$, become non-zero for later times and vanish again at $t = +\infty$. Although the fields are zero at $t = \pm\infty$, the potentials need not be, and the field that interpolates between two topologically different potentials for vanishing fields in the infinite past and the infinite future is called a Yang–Mills instanton. There are infinitely many topologically distinct 'vacua' with vanishing fields, just like the pendulum's minima at $\phi = 2\pi n, n = 0, \pm 1, \pm 2, ....$ A similar situation obtains for an electron in a periodic lattice potential with infinitely many different minima; because the electron can tunnel from one minimum to another, its ground state is a superposition of all the minima. This is an example of a Bloch wave.

the Yang–Mills case, we refer the reader to Huang [4]. In our problem, we start with the 'perturbative limit' $k^2 = 0$.

### 4.1 *The* $k^2 = 0$ *Case*

Since $K(k^2 = 0) = \pi/2$, we are in the small angle regime where $T = 2\pi\sqrt{l/g}$. Equation (9) then gives

$$T' = 2i\sqrt{\frac{l}{g}} \int_1^\infty \frac{\mathrm{d}y}{\sqrt{(y^2-1)}} = 2i\sqrt{\frac{l}{g}} K'(0) \quad . \qquad (15)$$

The integral is elementary, resulting in $K'(0) = 2\text{arccosh}(\infty) = \infty$. This is of course the degeneration of an elliptic function to a trigonometric one as the imaginary period goes to infinity. The duality relation then shows that $K(1)$, the real oscillation period of a pendulum starting at the top unstable equilibrium point $\phi = \pi$, is also infinite. The reason is simply that the motion cannot start without an initial perturbation, however small.

The real oscillation period of a pendulum starting at the top unstable equilibrium point $\phi = \pi$, is also infinite.

Let us next examine the actual function $\phi(t)$ in imaginary time, setting the energy $E$ (and thus also $k$) *strictly to zero* so that the pendulum starts at $\phi = 0$ in the infinite past. After canceling out overall constants, the zero-energy condition becomes

$$-(\dot{\phi})^2 + (2g/l)(1 - \cos\phi) = 0 \quad , \tag{14}$$

or, using half angles and taking square roots,

$$\dot{\phi} = \pm\frac{2g}{l}\sin\frac{\phi}{2} \quad . \tag{15}$$

Let us choose the plus sign. The solution will be a function of $t - t_0$, with $t_0$ an arbitrary constant which we set to zero, resulting in

$$\phi(t) = 4\arctan(\exp\omega t) \quad . \tag{16}$$

This is an example of a topological instanton [5], a semiclassical solution that tunnels in imaginary time. It starts from $t = -\infty$ at the original ground state $\phi = 0$, and ends up at the topologically distinct one $\phi = 2\pi$ at $t = +\infty$. The tunneling could also go in the opposite direction. This corresponds to taking the minus sign in (15) and might be called an anti-instanton. Thus the infinite period $T' = 2i\sqrt{l/g}K'(0)$ found in (13) can be viewed as the tunneling time of the instanton, which is defined as a classical solution interpolating in imaginary time between two ground states. The

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ground state of unquantized Yang–Mills theory is approximated by a Bloch wave-like [4] superposition of precisely such topologically inequivalent vacua connected by instantons, although the topological inequivalence of Yang–Mills vacua is not as intuitively graspable as our elementary example.

According to the Feynman path integral recipe the tunneling probability amplitude for the instanton is proportional to $\exp(iS/\hbar)$, where $S$ is the action for the entire path. The latter can be calculated exactly in imaginary time $it$. The result is a purely imaginary action, turning the amplitude to $\exp(-|S|/\hbar)$. We start with

$$|S| = |\int_{-\infty}^{+\infty} L(t)\mathrm{d}t| = \int_{-\infty}^{+\infty} \{\frac{1}{2}ml^2 \left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 + mgl (1-\cos\phi)\}\mathrm{d}t \ . \quad (19)$$

Using (14), one can eliminate the $\dot{\phi}^2$ term in favor of the potential energy, giving

$$|S| = 2mgl \int_{-\infty}^{+\infty} (1-\cos\phi)\ \mathrm{d}t \ . \quad (20)$$

Replacing $\mathrm{d}t$ by $\mathrm{d}\phi/(\mathrm{d}\phi/\mathrm{d}t)$, using half-angle formulas and remembering that $\phi(-\infty) = 0$ and $\phi(+\infty) = 2\pi$, the integral becomes

$$|S| = 2m\sqrt{gl^3} \int_0^{2\pi} \sin\frac{\phi}{2}\ \mathrm{d}\phi = 8m\sqrt{gl^3} \ . \quad (21)$$

Thus the instanton's contribution to the action is $\exp(-8m\sqrt{gl^3}/\hbar)$. It can be shown that the value of $|S|$ in (19) is a minimum (other than the trivial one $S = 0$) and therefore the instanton solution dominates the imaginary time Feynman path integral for the problem.

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### 4.2 *The $k = 1$ Case*

For completeness, let us also note that the result $K(0) = \pi/2$ for the small-angle simple pendulum implies $K(0) =$

**Box 3. The Action, WKB and Feynman's Path Integral**

Let us limit ourselves to one coordinate $x$ and the velocity $\dot{x}$. This is sufficient for discussing the essential concepts, and our problem happens to be one-dimensional in any case. The Lagrangian is $L = m\dot{x}^2/2 - U(x)$, where $m$ is the mass and $U$ is the potential energy of a non-relativistic particle. The action $S$ is given by $\int_0^t L\mathrm{d}t'$ and fixing the initial and final positions $x(0)$ and $x(t)$, one calculates an $S(t)$ for each conceivable path connecting these points. The classical path is found by extremizing $S$, which yields the equations of motion. In Feynman's path integral formulation of quantum mechanics, the quantity of interest is the probability amplitude, denoted by $< x(t), t|x(0), 0 >$, which, by Feynman's recipe, equals $A\Sigma \exp(iS[x(t)]/\hbar)$. The sum is over all paths and $A$ is a normalization constant. Each path thus weighs in with its phasor, i.e., a complex number of unit modulus. Since $S$ is stationary around the classical path, the phasors for nearby paths are nearly parallel. They add up to give a huge contribution, while the randomly oriented phasors from non-classical paths cancel each other out. For macroscopic objects, the angle $\alpha(S) \equiv (S_{\mathrm{cl}} - S)/\hbar$ represents a big deviation in the direction of the phasor from the classical; it will thus lead to destructive interference with other non-classical-path phasors. An elementary particle, on the other hand, can follow non-classical paths for which $\alpha(S)$ remains within a small angular range.

Of particular interest for our problem are tunneling paths where the momentum $p(x) = \sqrt{(E - U(x))/2m}$ is imaginary because $U > E$. When an exact quantum mechanical calculation of the tunneling probability for an arbitrary $U(x)$ is impracticable, the Wentzel–Kramers–Brillouin approach, invented independently by all three authors in the same year as the Schrödinger equation, provides an approximate answer. The method is based on adding up the phase changes $\exp(ip(x)\mathrm{d}x/\hbar)$ as the approximate plane wave moves in steps of length $\mathrm{d}x$. For a tunneling event with imaginary $p(x)$ from $x_1$ to $x_2$, this gives a transmission amplitude proportional to $\exp(-\int_{x_1}^{x_2} \sqrt{(E - U(x))/2m})\mathrm{d}x$. The relation of this expression to an also semi-classically evaluated Feynman recipe follows from putting $\int L\mathrm{d}t = \int (p\mathrm{d}x/\mathrm{d}t - E)\mathrm{d}t = \int p\mathrm{d}x - E\tau$, where $\tau$ is the transit time. Note that the semi-classical approach involves taking $E$ as constant during the passage and the first term is just the WKB expression. A detailed calculation (R Shankar, *Principles of quantum mechanics*, second edition, Kluwer Academic/Plenum publishers, New York, 1994.) shows that the factor $< x_2(t), t|x_1(0), 0 >$ actually includes another factor of $+E\tau$ in the exponential, leaving only the WKB term.

$K'(1) = \pi/2$ by the duality relation. Physically, this means that while the real time oscillations from $\phi_0 = -\pi$ to $\phi_0 = +\pi$ take forever, the (imaginary) tunneling time around the top point from $-\pi + \epsilon$ to $\pi - \epsilon$ is the same as the usual period of a small-angle pendulum.

If given a sufficiently high initial velocity, our pendulum can of course also complete full rotations in vertical circles in *a finite amount of time* (instead of swinging from $\phi_0 = -\pi$ to $\phi_0 = +\pi$ with the infinite period $K'(0)$).

The period of the full rotational motion can be made arbitrarily small by increasing the initial energy, and therefore cannot be identified with $K$ or $K'$ for any $k = \sin(\phi_0/2)$. Another way of achieving full rotations is of course to drive the pendulum with an external motor. Interestingly, if the familiar value $\omega = \sqrt{g/l}$ is chosen, there is neither compression nor tension in the rod at the topmost point.

## 5. Discussion: Tunneling and Imaginary Time in Nature

Quantum tunneling as a microscopic phenomenon is of course ubiquitous: The nitrogen atom tunnels back and forth across the equilateral triangle of hydrogen atoms in the ammonia ($NH_3$) molecule about $2.4\times10^{10}$ times per second; alpha particles tunnel through the repulsive Coulomb wall in nuclei and get out; a DC current flows across a thin insulating Josephson junction between two superconductors, to name a few familiar examples. It is now even the basis of some high technology devices such as tunnel diodes and scanning-tunneling microscopes. What we tried to show here is that a very familiar classical mechanical system also reveals connections with quantum mechanics and non-perturbative phenomena in Yang–Mills theories in an exact mathematical treatment, but of course the actual probability that a macroscopic pendulum will exhibit quantum tunneling is of course fantastically small. For example, for $m = 1$ kg and $l = 1$ m, $\exp(-8m\sqrt{gl^3}/\hbar)$ is of order $\exp(-10^{35})$! An appreciable probability is only possible if the argument of the exponential is of order unity. In SI units, this requires $l^{3/2} \sim (0.263 \times 10^{-34})/m$, which leaves a very narrow window for simultaneously physically meaningful values of $m$ and $l$. A length of 1 m leads to masses of the order of a billionth of an average atomic mass, while a mass of 1 kg produces a length nearly at the Planck scale of $10^{-34}$ m. On the other hand, if $l = 1$ nm, one finds $m \sim 10^{-22}$ kg. These last numbers are intriguingly within a few orders of magnitude of current nano- technology applications [6] involving simple harmonic oscillators, but considering the weakness of gravity relative to the competing forces at such small scales, it is very unlikely that any evidence for the 'imaginary time behavior' of a simple pendulum will ever be seen in a controlled experiment.

According to Hawking [7] and Vilenkin [8], however, the universe itself, the ultimate uncontrolled experiment and the ultimate macroscopic entity, may have started in imaginary time and switched to our usual time 'later' (whatever 'later' may mean in this setting)! In the Vilenkin version, the imaginary time arises in connection with the universe tunneling out of a state of zero energy just like the instanton here; indeed, the process is described by something called the 'de Sitter–Hawking–Moss instanton'. In the Hawking version, on the other hand, imaginary time is principally used to smooth out the Big Bang singularity. For more details and the debate about whether the approaches are equivalent, we refer the readers to Vilenkin's book.

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