Equations Governing Kepler's Laws of Planetary Motion

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1. Introduction

In the first lesson in astronomy, one is introduced to Kepler's Laws. There is a long history behind these laws and though they go by the name of Kepler, he stood on the shoulders of the giants who made observations for years before him, especially Tycho Brahe. A century after Kepler, Newton was able to derive Kepler's laws from his own laws of motion and his law of universal gravitation. Newton's second law of motion is concerned with the motion of objects subject to impressed forces. Newton's law of gravitation describes how masses attract each other through the force of gravity.

2. Statement of Kepler's Laws

Law 1. The orbit of every planet is an ellipse with the Sun at a focus. (See *Figure* 1.)

Law 2. A line joining a planet and the sun sweeps out equal areas during equal intervals of time.(See *Figure* 2.)

Law 3. The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

These laws can be derived from the following two laws:

Newton's second law: 'For a particle, the mass times the acceleration is equal to the force impressed on the particle'.

Newton's law of gravitation: 'Every object in the universe attracts every other object along a line joining the

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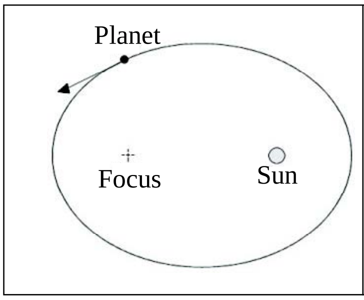

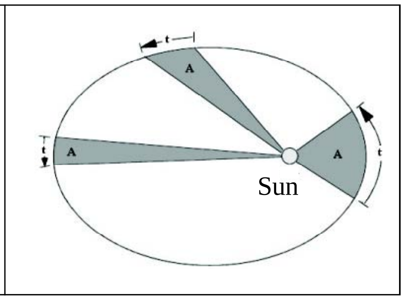

Figure 1 (left).
Figure 2 (right).

centres of the objects and is proportional to each object's mass and inversely proportional to the distance between the objects'.

3. Equations Governing Kepler's Laws

In a Cartesian coordinate system the position of a moving particle is given by $(x(t), y(t))$ at time t. Its velocity will be $(\dot{x}(t), \dot{y}(t))$. In polar coordinates, the position of a moving particle is $(r(t), \theta(t))$, where r is the radial distance from a fixed point and θ is the azimuthal angle measured from a fixed line.

If $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ denote the unit vectors in the radial direction and the tangential direction (perpendicular to the radial direction) pointing in the direction of rotation, respectively, then

$$\mathbf{r} = r\hat{\mathbf{r}}$$

$$\dot{\hat{\mathbf{r}}} = \dot{\theta}\hat{\boldsymbol{\theta}} \quad \text{and} \quad \dot{\hat{\boldsymbol{\theta}}} = -r\dot{\theta}\hat{\mathbf{r}}.$$

The velocity of a particle is given by

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

and the acceleration is

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}.$$

For constant distance r, $r\dot{\theta}^2$ is the centripetal acceleration towards the centre and for constant angular speed $\dot{\theta}$, $2\dot{r}\dot{\theta}$ is the Coriolis acceleration. Newton's law of gravitation prescribes the force of attraction of the sun on a planet as GMm/r^2, where M is the mass of the sun, m is the mass of the planet, G is the gravitational constant and r is the distance between the centres of the sun and the planet.

Newton's second law of motion then gives

$$m(\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} = -\frac{GMm}{r^2}\hat{\mathbf{r}}.$$

This gives two differential equations

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2} \tag{1}$$

and

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0. \tag{2}$$

Multiplying (2) by $\dot{r}$ and integrating, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = 0.$$

Therefore, $r^2\dot{\theta} = l, \quad$ a constant.

Substituting for $\dot{\theta}$ in equation (1), we get

$$\ddot{r} - r\frac{l^2}{r^4} = -\frac{GM}{r^2},$$

i.e.,

$$\ddot{r} = -\frac{GM}{r^2} + \frac{l^2}{r^3}. \tag{3}$$

Changing the independent variable from t to θ

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\cdot\dot{\theta} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\cdot\frac{l}{r^2} = -l\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{r}\right) = -l\frac{\mathrm{d}u}{\mathrm{d}\theta},$$

where

$$u = \frac{1}{r}.$$

Changing the dependent variable from r to u

$$\begin{aligned}\ddot{r} &= -l\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right) = -l\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t}\\ &= -l\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}\cdot\frac{l}{r^2} = -l^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}.\end{aligned}$$

Substituting in (3)

$$-l^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = -GMu^2 + l^2u^3$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \frac{GM}{l^2}.$$

If $u = (GM/l^2)v$, then we have

$$\frac{\mathrm{d}^2v}{\mathrm{d}\theta^2} + v = 1. \tag{4}$$

The solution to equation (4) is

$$v = 1 + A\cos(\theta - \theta_0)\,,$$

where A, θ_0 are arbitrary constants.

$$\frac{l^2}{GMr} = 1 + A\cos(\theta - \theta_0),$$

i.e.,

$$r(1 + A\cos(\theta - \theta_0)) = \frac{l^2}{GM} = p, \quad \text{say.} \tag{5}$$

Compare equation (5) to the polar coordinate equation of an ellipse, with the origin at a focus, namely,

$$r(1 + \varepsilon\cos\theta) = p = a(1 - \varepsilon^2), \tag{6}$$

where ε is the eccentricity of the ellipse, a is the semi-major axis of the ellipse and p is the semi-latus rectum of the ellipse. Measuring θ from the semi-major axis of the ellipse and r from the point on the ellipse nearest the focus, we have $\theta_0 = 0$ and $A = \varepsilon$ in (5). This proves Kepler's first law.

We have already seen that $r^2\dot{\theta} = l$, a constant. The rate at which a radius vector sweeps out area A at time t is given by

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r.r\dot{\theta} = \frac{1}{2}r^2\dot{\theta} = \frac{l}{2},$$

a constant. The area swept out from time t_1 to time t_2 is

$$\int_{t_1}^{t_2} \frac{\mathrm{d}A}{\mathrm{d}t} \cdot \mathrm{d}t = \frac{1}{2} l(t_2 - t_1).$$

The radius vector joining the planet and the Sun sweeps out equal areas in equal intervals of time, which is Kepler's second law.

In the case of an ellipse, the area covered in one orbit is

$$A = \pi a \cdot b,$$

where a is the semi-major axis and b is the semi-minor axis. In terms of the eccentricity ε, $b = a\sqrt{1-\varepsilon^2}$. The rate at which the radius vector sweeps out the orbit is $\mathrm{d}A/\mathrm{d}t = l/2$, a constant. So the period of an orbit is $P = A/\dot{A} = (2\pi a^2 \sqrt{1-\varepsilon^2}/l$. Therefore

$$\left(\frac{P}{2\pi}\right)^2 = \frac{a^4(1-\varepsilon^2)}{l^2} = \frac{a^4(1-\varepsilon^2)}{pGM} = \frac{a^4(1-\varepsilon^2)}{GM \cdot a(1-\varepsilon^2)} = \frac{a^3}{GM}.$$

$$P^2 \ \alpha \ a^3$$

This is Kepler's third law.

A detailed derivation of the laws and properties of ellipses can be found in Wikipedia, the free encyclopedia.

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Errata

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The Solar Physics Observatory at Kodaikanal and John Evershed (pp. 1032–1039): Due to an inadvertence on the part of the author, the article did not carry the credits to go with the photographs. The photographs appearing in the article were obtained from the IIA Archives. The error is regretted.

What is the Unit of Natural Selection (pp. 1047–1059): We regret that the author's name Ambika Kamath has been misspelt as Ambika Karanth.